Part A1: Differential Equations I

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Contents

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[*]based on notes by Peter Grindrod, Colin Please, Paul Tod, Lionel Mason and others

Introduction

The solution of problems in most parts of applied mathematics and some areas of pure can more often than not be reduced to the problem of solving some differential equations. Indeed, many parts of pure maths were originally motivated by issues arising from differential equations, including large parts of algebra and much of analysis. From Prelims, and even from school, you know how to solve some differential equations. Indeed most of the study of differential equations in the first year consisted of finding explicit solutions of particular ODEs or PDEs. However, for many differential equations which arise in practice one is unable to give explicit solutions and, for the most part, this course will consider what information one can discover about solutions without actually finding the solution. Does a solution exist? Is it unique? Does it depend continuously on the initial data? How does it behave asymptotically? What is appropriate data?

So, first we will develop techniques for proving Picard's theorem for the existence and uniqueness of solutions of ODEs; then we will look at how phase plane analysis enables us to estimate the long term behaviour of solutions of plane autonomous systems of ODEs. We will then turn to PDEs and show how the method of characteristics reduces the solution of a first order semi-linear PDE to solving a system of non-linear ODEs. Finally we will look at second order semi-linear PDEs: We classify them and investigate how the different types of problem require different types of boundary data if the problem is to be well posed. We then look at how the maximum principle enables us to prove uniqueness and continuous dependence on the initial data for two very special problems: Poisson's equation and the inhomogeneous heat equation – each with suitable data.

Throughout, we shall use the following convenient abbreviations: we shall write

DEs: for differential equations.

ODEs: for ordinary DEs, i.e. differential equations with only ordinary derivatives.

PDEs: for partial DEs, i.e. differential equations with partial derivatives.

The course contains four topics, with a section devoted to each. The chapters are:

1. ODEs and Picard's Theorem (for existence/uniqueness of solutions/continuous dependence on initial data).

2. Plane autonomous systems of ODEs

3. First order semi-linear PDEs: the method of characteristics.

4. Second-order semi-linear PDEs: classification; well posedness; the Maximum Principle and its consequences

Books

The main text is P J Collins *Differential and Integral Equations*, O.U.P. (2006), which can be used for the whole course (Chapters 1-7, 14, 15).

Other good books which cover parts of the course include

W E Boyce and R C DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 7th edition, Wiley (2000).

E Kreyszig, *Advanced Engineering Mathematics, 8th Edition*, Wiley (1999).

G F Carrier and C E Pearson, *Partial Differential Equations – Theory and Technique*, Academic (1988).

J Ockendon, S Howison, A Lacey and A Movchan, *Applied Partial Differential Equations*, Oxford (1999) [a more advanced text].

PART I Ordinary Differential Equations

1   ODEs and Picard's Theorem

1.1   What is an ODE?

An ODE is an equation for $y(x)$ of the form

$$G(x, y, y', y'', \ ..., \ y^{(n)}) = 0.$$

We refer to y as the dependent variable and x as the independent variable. Usually this can be solved for the highest derivative of y and written in the form

$$\frac{d^n y}{dx^n} := y^{(n)}(x) = F(x, y, y', \ ..., \ y^{(n-1)}).$$

Then the *order* of the ODE is n, the order of the highest derivative which appears. Given an ODE, certain obvious questions arise. We could ask:

- Does it have solutions? Can we find them (explicitly or implicitly)? If not, can we at least say something about their qualitative behaviour?

- Given data e.g. the values $y(a)$, $y'(a)$, ... of $y(x)$ and its first $n - 1$ derivatives at some initial value a of x, does it have a solution? is it unique? does it depend continuously on the given data?

We shall consider these questions in Part I.

For simplicity, we begin with a first-order ODE with data:

$$y'(x) = f(x, y(x)) \text{ with } y(a) = b. \tag{1.1}$$

This is an *initial value problem* or IVP, since we are given y at an initial, or starting, value of x . You know how to solve a variety of equations like this. You might expect that a solution *exists*, so there is some function that satisfies the equations (even if you cannot find a formula for it) and perhaps you expect, for a given initial values, the solution is *unique* (there is only one function that satisfies the ODE and the initial data) as you may not have encountered the following difficulties.

1.2   Warning examples:

Consider this IVP:

$$y' = 3y^{2/3}; \quad y(0) = 0. \tag{1.2}$$

So separate the variables (a prelims technique):

$$\int \frac{dy}{3y^{2/3}} = \int dx,$$

to get $y = (x + A)^3$, so if $y(0) = 0$

  (i) There is a solution $y = x^3$;

 (ii) But evidently there is another solution: by direct checking $y = 0$ will do;

(iii) In fact we can find that there are infinitely many solutions. Pick a, b with $a \leq 0 \leq b$ and define

$$\begin{aligned}
y &= (x - a)^3 & x < a \\
&= 0 & a \leq x < b \\
&= (x - b)^3 & b \leq x
\end{aligned}$$

The solution does exist but is not unique (in fact far from it, since we've found infinitely many solutions).

Furthermore, even if a solution of (1.1) exists, it may not exist for all x.

For example consider the IVP

$$y' = y^2; \quad y(0) = 1. \tag{1.3}$$

Using separation of variables we can see this has solution $y = \dfrac{1}{1 - x}$; so $y \to \infty$ as $x \to 1$, and the solution only exists on $x < 1$. We will see later that this solution is in fact unique.

So, if we want to have a unique solution to the problem (1.1) we must impose conditions on f, and we cannot necessarily expect to have solutions for all x. This will be the first *existence theorem* which you've encountered.The proof is quite technical, certainly the most technical thing in the course. In particular, you need to remember from Prelims the *Weierstrass M-test* for convergence of a series of functions.

To be precise then, we shall seek a solution of problem (1.1) in a rectangle R about the initial point $(x, y) = (a, b)$, so suppose $R = \{(x, y) : |x - a| \leq h, \ |y - b| \leq k\}$ as in figure 1.1:
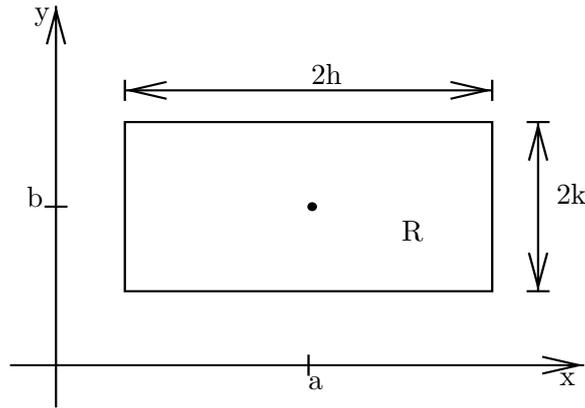
Figure 1.1: The rectangle R

Our first assumption is that $f(x, y)$ is continuous in R.

Now integrate (1.1) from a to variable x:

$$[y(t)]_a^x = \int_a^x f(t, y(t)) dt = y(x) - y(a)$$

so rearranging

$$y(x) = b + \int_a^x f(t, y(t)) dt. \tag{1.4}$$

We have transformed the differential equation to an *integral equation* - the unknown y is given in terms of an integral rather than a differential. There is a general theory of these, but we only need to deal with the particular case of (1.4). The standard approach is to seek a solution by *iteration* or successive approximation.

1.3 Picard's method of successive approximation

We start with an initial guess and then improve it. The guesses, or successive approximations or iterates, are labeled $y_n(x)$ starting with $y_0(x)$. Take

$$\left. \begin{array}{l} y_0(x) = b \\ y_{n+1}(x) = b + \int_a^x f(t, y_n(t)) dt \end{array} \right\} \tag{1.5}$$

That is, we start with the simplest guess, that y equals its initial value, and at each stage substitute the current guess into the right-hand-side of (1.4) to get the next guess. We need to know if this process converges, and if it does whether it converges to a solution of the problem (1.4). Consider the differences between successive approximations:

$$\left. \begin{array}{l} e_0(x) = b \\ e_{n+1}(x) = y_{n+1}(x) - y_n(x) \end{array} \right\} \tag{1.6}$$

8

and note that

$$y_n(x) = \sum_0^n e_k(x). \tag{1.7}$$

We want y_n to converge, so we want the series $\sum_0^n e_k(x)$ to converge. So we must estimate the differences $e_n(x)$. This will need a condition on f, but here is the key idea.

Notice that

$$e_{n+1}(x) = y_{n+1}(x) - y_n(x)$$
$$= \int_a^x [f(t, y_n(t)) - f(t, y_{n-1}(t))]dt$$

and recall that *the modulus of the integral of a function is less than or equal to the integral of the modulus* (because the function can be negative). Therefore

$$|e_{n+1}(x)| \leq \left| \int_a^x |f(t, y_n) - f(t, y_{n-1})| \, dt \right|. \tag{1.8}$$

(The modulus outside the integral on the right hand side is required to cover the case $x \leq a$.) Recall we want the $y_n = \sum e_n$ to converge. This motivates the definition:

1.4 The Lipschitz condition

A function $f(x, y)$ continuous on a rectangle R satisfies a *Lipschitz condition* with constant L if $\exists$ real positive L such that

$$|f(x, u) - f(x, v)| \leq L|u - v| \text{ for } (x, u) \in R, \ (x, v) \in R. \tag{1.9}$$

This is a new condition on a function, stronger than being continuous but weaker than being differentiable. It turns out to be the right condition to make Picard's theorem, which is the existence theorem we want, work.

Important note: One way to ensure that f is Lipschitz continuous on R is the following: Suppose that, on R, f is differentiable with respect to y , with $|f_y(x, y)| \leq K$. Then for any $(x, u) \in R, \ (x, v) \in R$ the mean value theorem gives

$$|f(x, u) - f(x, v)| = |f_y(x, w)(u - v)| \leq K|u - v| \tag{1.10}$$

where w is some intermediate value . So, f is clearly Lipschitz on such intervals.

On the other hand $f(y) = |y|$ is Lipschitz continuous, but is not differentiable at $y = 0$.

1.5 Picard's Theorem

Theorem 1.1. (Picard's existence theorem): *The IVP (1.1)*

$$y' = f(x, y) \ \text{ with } y(a) = b.$$

has a solution in the rectangle $R := \{(x, y) : |x - a| \le h, |y - b| \le k\}$ *provided:*

P(i): *(a) f is continuous in R, with bound M (so $|f(x, y)| \le M$) and (b) $Mh \le k$.*
P(ii): *f satisfies a Lipschitz condition in R.*

Furthermore, this solution is unique.

Proof

We want to prove that y_n converges as $n \to \infty$ and will do this by showing that the series in (1.7) converges. We break the proof into a series of steps:

(i) Each $y_n(x)$ is continuous and its graph lies inside R.

The graph property means that if $|x - a| \le h$ then $(x, y_n(x)) \in R$, so the graph of y_n goes right across R (it 'comes out the side not the top or bottom'). It is needed so that P(i) and P(ii) hold for $f(x, y_n(x))$. So we must show that if $|x - a| \le h$ then $|y_n(x) - b| \le k$. Assume that for $|x - a| \le h$ $y_n(x)$ is continuous and $|y_n(x) - b| \le k$. Then for $|x - a| \le h$, by the continuity of f and the properties of integration, $y_{n+1}(x)$ is continuous, and by P(i),

$$|y_{n+1}(x) - b| \le \left| \int_a^x |f(t, y_n(t))| dt \right|$$

$$\le \left| M \int_a^x dt \right| = M|x - a| \le Mh \le k.$$

But this is trivially true for y_0, so the result is true by induction.
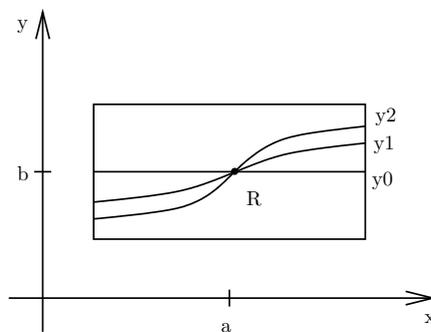
Figure 1.2: successive iterates graphed in R.

(ii) The graph property and the Lipschitz condition P(ii) imply that, if $|t - a| \le h$, then for some positive L,

$$|f(t, y_n(t)) - f(t, y_{n-1}(t))| \le L|y_n(t) - y_{n-1}(t)|. \tag{1.11}$$

10

(iii) We next prove that for $|x - a| \le h$:

$$|e_n(x)| \le \frac{L^{n-1}M}{n!}|x - a|^n \tag{1.12}$$

$$\le \frac{L^{n-1}M}{n!}h^n. \tag{1.13}$$

From (1.8) and (1.11):

$$|e_{n+1}(x)| \le \left| \int_a^x |f(t, y_n(t)) - f(t, y_{n-1}(t))|dt \right|$$

$$\le L \left| \int_a^x |y_n(t) - y_{n-1}(t)|dt \right|$$

and so

$$|e_{n+1}(x)| \le L \left| \int_a^x |e_n(t)|dt \right|. \tag{1.14}$$

Now we prove (1.12) by induction:

$$e_1(x) = y_1(x) - b = \int_a^x f(t, b)dt. \tag{1.15}$$

By **P(i)**, f is bounded by M so that

$$|e_1(x)| \le \left| \int_a^x |f(t, b)|dt \right| \le M|x - a|, \tag{1.16}$$

so (1.12) is true for $n = 1$. Now suppose that (1.12) is true for n, then

$$|e_{n+1}(x)| \le L \left| \int_a^x |e_n(t)|dt \right|$$

$$\le L \left| \int_a^x \frac{L^{n-1}M}{n!}|t - a|^n dt \right| = \frac{L^n M}{(n+1)!}|x - a|^{n+1},$$

so that (1.12) is true by induction. (1.13) is immediate.

(iv) The series (1.7), and hence the sequence y_n, converges uniformly to a continuous function.

The uniform convergence of the series is immediate from (1.13), using the Weierstrass M-test, since the series with n-th term $M_n = \dfrac{L^{n-1}Mh^n}{n!}$ converges and is independent of x. Thus, because $y_n(x) = \sum_0^n e_k(x)$,

$$y_n(x) \to y_\infty(x), \text{ say, uniformly.}$$

The continuity of $y_\infty(x)$ then follows from the continuity of the $y_n(x)$ (as the uniform limit of continuous functions is continuous).

So the iteration has a limit, which is a continuous function.

11

(v) y_∞ is a solution of (1.1).

We can take the limit in (1.5) to find

$$y_\infty(x) = b + \int_a^x f(t, y_\infty(t))dt. \qquad (1.17)$$

This needs justification: From the Lipschitz continuity of f

$$|f(x, y_n(x)) - f(x, y_\infty(x))| \leq L|y_n(x) - y_\infty(x)|$$

so that $f(x, y_n(x))$ converges uniformly to $f(x, y_\infty(x))$. Thus we may take the limit under the integral sign. As $f(x, y_\infty(x))$ is continuous, the RHS in (1.17) is differentiable; therefore, so is $y_\infty(x)$. Differentiate (1.17) to find

$$y_\infty'(x) = f(x, y_\infty(x))$$

and since also $y_\infty(a) = b$, y_∞ is a solution of the IVP.

(vi) Uniqueness: if $y(x)$ and $Y(x)$ are two solutions of the problem, consider their difference:

$$e(x) = y(x) - Y(x) = \int_a^x (f(t, y(t)) - f(t, Y(t)))dt$$

$$\text{so } |e(x)| \leq \left| \int_a^x |f(t, y(t)) - f(t, Y(t))|dt \right|$$

$$\leq L \left| \int_a^x |y(t) - Y(t)|dt \right|$$

$$\leq L \left| \int_a^x |e(t)|dt \right| \qquad (1.18)$$

where we have used the Lipschitz condition. Now $e(x)$ is continuous on R; therefore, it is bounded say $|e(x)| \leq B$ so

$$|e(x)| \leq L \left| \int_a^x Bdt \right| = LB|x - a|$$

and inducting on n, using 1.18, for each n

$$|e(x)| \leq BL^n \frac{|x - a|^n}{n!}$$

So that for each n

$$|e(x)| \leq B\frac{L^n h^n}{n!} \to 0 \text{ as } n \to \infty \text{ and } e(x) = 0.$$

Thus the difference is zero, so the solutions are the same i.e. unique (uniqueness proofs almost always go like this: assume there are two and make their difference vanish).

12

This ends the proof of Picard's Theorem. □

Since the warning example doesn't have a unique solution, something goes wrong for it. As an exercise, show that the warning example fails the Lipschitz condition (in any neighbourhood of the initial point).

The following example also fails the Lipschitz condition in any neighbourhood of the initial point $y = 0$. However, the Lipschitz condition does hold on any rectangle which does not contain any point $(x, 0)$.

Example: Consider the IVP

$$y' = x^2 y^{1/5}, \quad y(0) = b,$$

(so $f(x, y) = x^2 y^{1/5}$).

(Note that when we write $y^{1/5}$ here we mean to take the the real root: so that if y is negative we will take $-|y|^{1/5}$.)

Case $b = 0$: $f(x, y)$ does not satisfy a Lipschitz condition on any rectangle of the form $R_0 = \{(x, y) : |x| \le h, \; |y| \le k\}$, where $h > 0$ and $k > 0$.

Suppose it does, then there exists a finite constant L such that for all $|x| \le h$ and $|y| \le k$

$$|x^2| |y^{1/5} - 0| \le L|y - 0|$$

so in particular

$$|h^2| |y^{-4/5}| \le L.$$

But this is a contradiction as $|h^2| |y^{-4/5}|$ is unbounded as $y \to 0$ so the function is not Lipschitz continuous on R_0. So Picard's theorem will fail if we take $b = 0$. (In section 1.4 we saw that f is Lipschitz continuous on any rectangle where its derivative with respect to y exists and is bounded. The problem here is that the derivative of f is unbounded as $y \to 0$ – indeed the derivative does not exist at $y = 0$.)

Case $b > 0$: However, Picard's theorem will be satisfied if we take as initial condition $y(0) = b > 0$, provided we take a rectangle, R_b, given by $R_b = \{|x| \le h, \; |y - b| \le k\}$ when $0 < k < b$, so that y cannot be zero in this rectangle .

On any such rectangle $f_y(x, y) = \frac{x^2 y^{-4/5}}{5}$ is bounded, so, by (1.10), f is Lipschitz continuous, and **P(ii)** is satisfied.

For **P(i)**: $f(x, y)$ is continuous on R_b and

$$\max_{R_b} |x^2 y^{1/5}| \le M = h^2 (b + k)^{1/5},$$

so Picard's theorem applies in a rectangle where $h > 0$ satisfies

$$h^2 (b + k)^{1/5} h \le k.$$

That is

$$h^3 \le \frac{k}{(b + k)^{1/5}}. \tag{1.19}$$

We can of course solve this problem directly if we wish

$$y = \left(4x^3/15 + b^{4/5}\right)^{5/4},$$

so actually the solution exists for all x. Note the solution above is valid for $b = 0$ BUT the trivial solution is also valid.

Remark: Note that the proof of uniqueness of the solution only holds among those solutions whose graph lies in R. However, actually, if **P(i)** and **P(ii)** hold, then, for $|x - a| \le h$, the graph of any solution of (1.1) must lie in R.

Indeed, suppose not. Then, by the continuity of y, there will exist a 'first' x_0, with $|x_0 - a| < h$, where $(x_0, y(x_0))$ is on the boundary of R. That is such that $|x_0 - a| < h$, $|y(x_0) - b| = k$ but $|y(x) - b| < k$ if $|x - a| < |x_0 - a|$, see figure 1.3. But then

$$|y(x_0) - b| \le \left| \int_a^{x_0} |f(s, y(s))| ds \right| \le M|x_0 - a| < Mh = k$$

a contradiction.

Figure 1.3:

14
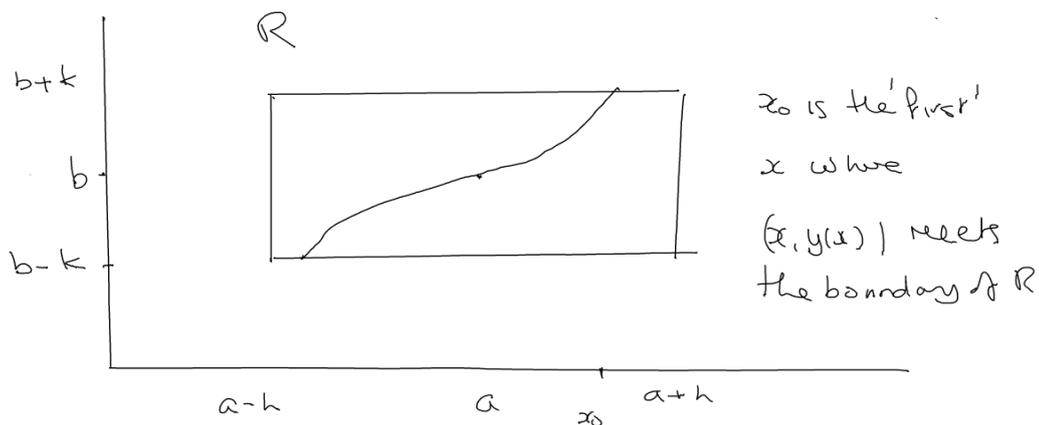
1.6 Extension of solutions and global existence.

The result in Section 1.5 is a *local* result. The solution is only valid in $[a-h, a+h]$, where h satisfies $Mh \leq k$ (though this h need not be the best possible).

As we have seen there are examples of initial value problems where solutions do not exist for all $x \in (-\infty, \infty)$. So we would like to find conditions which guarantee that either the solution does exist for $x \in (-\infty, \infty)$ or at least for a maximum interval. One such condition is the *global Lipschitz condition*, where we can find a constant L such that the Lipschitz condition holds for all y. First we will see that if the Lipschitz condition is global in y, but L still depends on the interval $[a-h, a+h]$, then there is existence on all of $[a-h, a+h]$

Suppose we require that $f(x, y)$ is continuous for all $x \in [a-h, a+h]$ and instead of (**P(ii)**) we have

(**P(iii)**): $f(x, y)$ satisfies the Lipschitz condition for all real y and all $x \in [a-h, a+h]$.

Then the condition $Mh \leq k$ is not required. If we investigate the proof of the Picard existence theorem, we see that M is required also in (1.16). However there it is sufficient to take $M = \sup_{x \in [a-h, a+h]} |f(x, b)|$, which exists as $f(x, b)$ is continuous on the closed bounded interval $[a-h, a+h]$. So that the solution exists and is unique $\forall x \in [a-h, a+h]$.

But now we can see that if (**P(iii)**) holds for each $h > 0$, the the solution exists for all x. In this case we say that we have a *global solution*.

Example: If $f(x, y) = p(x)y + q(x)$, where p and q are continuous on $|x-a| \leq h$, then f satisfies (**P(iii)**).

Remark We do not need the interval in (**P(iii)**) to be a balanced interval (ie of the form $[a-h, a+h]$) because we can deal with $x \leq a$ and $x \geq a$ separately. Thus if $a \in [c, d]$ and we require $f(x, y)$ is continuous for $x \in [c, d]$ and satisfies the Lipschitz condition for all real y and all $x \in [c, d]$, then a solution of (1.1) exists and is unique for all $x \in [c, d]$.

1.7 Gronwall's inequality and continuous dependence on the initial data.

We will now prove Gronwall's inequality which will be used provide another proof of uniqueness of solutions, but will also be used to show that solutions depend continuously on the initial data.

Theorem 1.2. (Gronwall's inequality) : *Suppose $A \geq 0$ and $b \geq 0$ are constants and v is a non-negative continuous function satisfying*

$$v(x) \leq b + A \left| \int_a^x v(s) ds \right| \tag{1.20}$$

then

$$v(x) \leq b e^{A|x-a|}.$$

(The modulus is needed to take care of the case $x \leq a$.)

Proof: We use an integrating factor.

For $x \geq a$ let $V(x) = \int_a^x v(s)ds$, so that $V'(x) = v(x)$ and we have

$$V'(x) \leq b + AV(x).$$

Multiply through by the integrating factor e^{-Ax} so

$$
\begin{aligned}
(V'(x) - AV(x))e^{-Ax} &\leq be^{-Ax} \text{ that is} \\
\frac{d}{dx}(V(x)e^{-Ax}) &\leq be^{-Ax}, \text{ so, integrating and noting that } V(a) = 0 \\
V(x)e^{-Ax} &\leq \int_a^x be^{-As}ds = \frac{b}{A}(e^{-Aa} - e^{-Ax}), \text{ so} \\
V(x) &\leq \frac{b}{A}(e^{A(x-a)} - 1).
\end{aligned}
$$

Finally, using (1.20)

$$v(x) \leq b + A\int_a^x v(s)ds = b + AV(x) \leq b + A\frac{b}{A}(e^{A(x-a)} - 1) = be^{A(x-a)},$$

as required. Similarly if $x \leq a$. $\qquad\square$

Remark: Gronwall's inequality says that v is bounded above by the solution of the integral equation one obtains when there is equality in (1.20). For, if we differentiate

$$v(x) = b + A\int_a^x v(s)ds,$$

we get

$$v'(x) = Av(x), \quad v(a) = b,$$

which has solution $v(x) = be^{A(x-a)}$.

Suppose now that y and z are solutions of the ordinary differential equation $y'(x) = f(x, y(x))$ with $y(a) = b$ and $z(a) = c$, where f satisfies conditions (**P(i)**) and (**P(ii)**). Then

$$y(x) - z(x) = b - c + \int_a^x (f(s, y(s)) - f(s, z(s)))ds$$

so that

$$|y(x) - z(x)| \leq |b - c| + \left| \int_a^x L|y(s) - z(s)| ds \right|$$

and by Gronwall's inequality

$$|y(x) - z(x)| \leq |b - c| e^{L|x-a|} \leq |b - c| e^{Lh}. \tag{1.21}$$

Thus we have related $|y(x) - z(x)|$ to $|b - c|$.

We say a solution is *continuously dependent on the initial data* if we can make $|y(x) - z(x)|$ as small as we like by taking $|b - c|$ small enough. In other words the error in the solution will be small provided the error in the initial data is small enough. To be precise, in this case, solutions are continuously dependent on the initial data for $x \in [a - h, a + h]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if y and z are as above,

$$|b - c| < \delta \quad \Rightarrow |y(x) - z(x)| \leq \epsilon, \quad \forall x \in [a - h, a + h].$$

This is clearly true from (1.21), because given $\epsilon > 0$, we have $|y(x) - z(x)| \leq \epsilon$ whenever $|b - c| < e^{-Lh}\epsilon$, so we can take $\delta = e^{-Lh}\epsilon$.

We could also use this to prove uniqueness. For if $b = c$, then $y(x) = z(x)$.

1.8 Picard's Theorem via the CMT

We can prove Picard's theorem in a more efficient way, which is really equivalent to our previous method, by using the contraction mapping theorem (CMT). This is a very useful method of proving existence and uniqueness of solutions of nonlinear differential equations and many, many other things besides. The results we need will be discussed in the course on Metric Spaces and Complex Analysis. We will assume the results proved there.

Define $\mathcal{C}_h = \mathcal{C}([a - h, a + h]; [b - k, b + k])$, the space of continuous functions mapping $[a - h, a + h]$ to $[b - k, b + k]$. As is shown in the Metric Spaces course, for $u, v \in \mathcal{C}_h$ if we define

$$d(u, v) := ||u - v||_{\sup} := \sup_{x \in [a-h,a+h]} |u(x) - v(x)|$$

then $(\mathcal{C}_h, d)$ is a complete metric space (we call d the "sup norm").

Also we say that $T : \mathcal{C}_h \to \mathcal{C}_h$ is a contraction if there exists $K < 1$ such that

$$||T(u) - T(v)||_{\sup} \leq K||u - v||_{sup},$$

and then we have the CMT, which when applied to $\mathcal{C}_h$ says:

Theorem 1.3. (Contraction Mapping Theorem) *(Banach) Let $T : \mathcal{C}_h \to \mathcal{C}_h$ be a contraction on the complete metric space $\mathcal{C}_h$. Then there is a unique fixed point $y \in \mathcal{C}_h$ such that $Ty = y$.*

Now we want to prove Picard's Theorem via the CMT. In this case we first show that a unique solution exists for $|x - a| \leq \eta$, where η is small enough. Then if $Mh \leq k$ we can extend to $|x - a| \leq h$.

We again consider the IVP (1.1)

Theorem 1.4. (Picard's existence theorem.) *Suppose that conditions* $\mathbf{P(i)}(a)$ *and* $\mathbf{P(ii)}$ *hold. Then there exists* $\eta \leq h$, *such that the initial value problem (1.1) has a unique solution for* $x \in [a - \eta, a + \eta]$.

Proof.

The strategy is to express (1.1) as a fixed point problem and use the CMT.

As before, we can write the initial value problem as an integral equation

$$y(x) = b + \int_a^x f(s, y(s))ds \qquad (1.22)$$

Provided $f(s, y(s))$ is continuous in s, y is a solution of the differential equation if and only if y is a solution of the integral equation.

If we define

$$(Ty)(x) = b + \int_a^x f(s, y(s))ds$$

then we can write (1.22) as a fixed point problem

$$y = Ty.$$

We will work in the complete metric space $\mathcal{C}_\eta = \mathcal{C}([a - \eta, a + \eta]; [b - k, b + k])$, where we will choose $\eta \leq h$ so that $T : \mathcal{C}_\eta \to \mathcal{C}_\eta$ and is a contraction.

(i) To prove that $T : \mathcal{C}_\eta \to \mathcal{C}_\eta$: First we note that from the properties of integration, $(Ty)(x) \in \mathcal{C}([a - \eta, a + \eta]; \mathbb{R})$.

Now we require that $||Ty - b||_{\sup} \leq k$ if $||y - b||_{\sup} \leq k$. But

$$||Ty - b||_{\sup} = \sup_{x \in [a-\eta,a+\eta]} \left| \int_a^x f(s, y(s))ds \right| \qquad (1.23)$$

$$\leq \sup_{x \in [a-\eta,a+\eta]} \left| \int_a^x |f(s, y(s))|ds \right| \qquad (1.24)$$

$$\leq M\eta \leq k, \qquad (1.25)$$

provided $M\eta \leq k$.

(ii) To show that T is a contraction for suitable η:

$$
\begin{aligned}
||Tu - Tv||_{\sup} &= \sup_{x \in [a-\eta, a+\eta]} \left| \int_a^x f(s, u(s)) - f(s, v(s)) ds \right| \\
&\leq \sup_{x \in [a-\eta, a+\eta]} \left| \int_a^x |f(s, u(s)) - f(s, v(s))| ds \right| \\
&\leq \sup_{x \in [a-\eta, a+\eta]} \left| \int_a^x L|u(s) - v(s)| ds \right| \leq L\eta ||u - v||_{\sup} \leq K ||u - v||_{\sup}
\end{aligned}
$$

where $K < 1$ provided $\eta < 1/L$. Choose $\eta < \min\{h, k/M, 1/L\}$. Then T satisfies the conditions of the CMT and has a unique fixed point, $y(x)$, which is continuously differentiable and is the unique solution of the initial value problem. $\qquad\square$

Note that our proof using CMT produces a more restricted range of x values than did our proof on one dimension. The range of η depends on L as well as M and k. However, if $Mh \leq k$, actually we only need $\eta \leq h$, and we can now extend the range of the solution to all $x \in [a-h, a+h]$, by iteration.

Corollary 1.5. *If $Mh \leq k$, then there is a unique solution for all $x \in [a-h, a+h]$*

Proof: We look at $x \geq a$ first. If $h < 1/L$ we are done. (Take $\eta = h$.)

Otherwise choose $\eta_1 < 1/L$.

Then, from Theorem 1.4, there exists a unique solution, y_0 say, on $[a, a+\eta_1]$.

Now choose $\eta_2 = \min\{2\eta_1, h\}$, and look for a solution, y_1 say, on $[a+\eta_1, a+\eta_2]$, of the ODE with initial data $y_1(a+\eta_1) = y_0(a+\eta_1)$.

Now define
$$
\begin{aligned}
y(x) &= y_0(x), x \in [a, a+\eta_1] \\
y(x) &= y_1(x), x \in [a+\eta_1, a+\eta_2]
\end{aligned}
$$

To constuct y_1: As in Theorem 1.4, but we now work in $\mathcal{C}_1 := \mathcal{C}([a+\eta_1, a+\eta_2]; [b-k, b+k])$, and take (for $a+\eta_1 \leq x \leq a+\eta_2$)

$$
\begin{aligned}
(T_1 y)(x) &= y_0(a+\eta_1) + \int_{a+\eta_1}^x f(s, y(s)) ds \\
&= b + \int_a^{a+\eta_1} f(s, y_0(s)) ds + \int_{a+\eta_1}^x f(s, y(s)) ds. \qquad (1.26)
\end{aligned}
$$

So $T_1 : \mathcal{C}_1 \to \mathcal{C}_1$ because from (1.26)

$$
||T_1 y - b||_{\sup} \leq M\eta_1 + M(x - (a+\eta_1)) = M(x-a) \leq M\eta_2
$$

$$
\leq Mh \leq k.
$$

Also T_1 is a contraction as in Theorem 1.4 because $\eta_2 - \eta_1 < 1/L$. Thus there is a unique solution y_1 say.

So y is a solution of (1.1) for $x \in [a, a + \eta_2]$.

We can repeat this process both to the left and the right to get a unique solution valid on $[a - h, a + h]$ as required. $\qquad\square$

Corollary 1.5
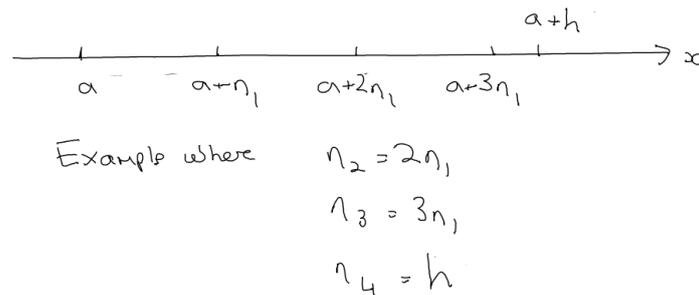
Figure 1.4: Corollary 1.5, where $\eta_2 = 2\eta_1$, $\eta_3 = 3\eta_1$, $\eta_4 = h$.

Global Existence: If for $x \in [a - h, a + h]$, and all y, $f(x, y)$ is continuous and is Lipschitz continuous in y (condition **P(iii)**), then we can work in $\mathcal{C}_h = \mathcal{C}([a-h, a+h]; \mathbb{R})$, so, as before, the condition $Mh \le k$ is no longer needed and we have existence on all of $[a - h, a + h]$. Furthermore, in this case the proof of Corollary 1.5 is simpler because we no longer require $||T_1 y - b||_{\sup} \le k$ etc.

Comparison of the two methods:

(1) The proof using the CMT is shorter and simpler than the direct proof because much of the work has been done in proving the CMT and once we have chosen a suitable space, we have only to check that the conditions apply. Also, the CMT automatically gives uniqueness of solutions, which has to be proved separately in the direct method.

2) By Theorem 8.2 of the Prelims Analysis II lecture notes, a sequence of continuous functions y_n converges in the sup norm if and only if it is uniformly convergent. Thus convergence in the sup norm is equivalent to uniform convergence. Furthermore in the CMT the fixed point is given by the limit in $\mathcal{C}_h$ of $y_n = T y_{n-1}$, with y_0 any point in the space. So if we take $y_0(x) = b$ (for all x) the fixed point is given by the uniform limit of the successive approximations as in the direct proof.

20

(3) One feature of the proof using the CMT was that it produced a less delicate result, in that the range of x for which it applied is more restricted. (Though it was easy to extend the range using iteration.) This sometimes happens when we use abstract results rather than direct computations, because the direct computations can be more delicate. We can see why this happens in this case if we investigate the direct proof. In the direct proof we are working pointwise and each time we apply the inductive step using (1.14) we integrate $(x-a)^n$, and thus end up dividing by $n!$, so we have a series which converges for all $x \in [a-h, a+h]$. But in the CMT we are working in $\mathcal{C}([a-h, a+h])$, so on each integration we take the supremum which, of course, does not depend on x, and thus we integrate a constant, so the $n!$ is absent.

Remark: There are many other fixed point theorems, and other abstract results, which can be used to prove existence of solutions of more general equations involving derivatives (including partial derivatives) and integrals. These powerful theorems generally require some general theory of Banach and Hilbert spaces (see the B4 courses) and a knowledge of suitable spaces (eg Sobolev spaces) in which to apply them (see the various C4 courses).

1.9 Picard's Theorem for systems via the CMT

We now want to look at existence and uniqueness of solutions of systems of ODEs. As well as being of interest in itself, this will be useful in particular for proving the existence of solutions of equations with higher order derivatives. We consider a pair of first order ODEs, for the functions y_1 and y_2.

$$y_1'(x) = f_1(x, y_1(x), y_2(x)) \tag{1.27}$$
$$y_2'(x) = f_2(x, y_1(x), y_2(x)) \tag{1.28}$$
with initial data $\quad y_1(a) = b_1, \quad y_2(a) = b_2. \tag{1.29}$

We can introduce vector notation

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

So we can write equations (1.27)–(1.29) as

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x)), \tag{1.30}$$
$$\underline{y}(a) = \underline{b}, \tag{1.31}$$

Now we want to prove Picard's Theorem for such systems of ODEs. Our previous proof using the CMT will extend in a very natural way.

We need a 'distance' in $\mathbb{R}^2$. In the Metric Spaces course the various norms l^1, l^2 (the Euclidean distance) and l^∞ on $\mathbb{R}^n$ were defined. We could use any of these (or any other norm on R^n), but we will make the fairly arbitrary choice to use the l^1 norm,

$||\underline{y}||_1 = |y_1| + |y_2|$. In place of the rectangle R we will use the subset $S = \{(x, \underline{y}) \in \mathbb{R}^3 : |x-a| \le h,\ \underline{y} \in B_k(\underline{b})\}$, where $B_k(\underline{b})$ is the closed disc in $\mathbb{R}^2$ centred on $\underline{b}$, radius k. That is $B_k(\underline{b}) = \{\underline{y} \in \mathbb{R}^2 : ||\underline{y} - \underline{b}||_1 \le k\}$ (ie $B_k(\underline{b}) = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - b_1| + |y_2 - b_2| \le k\}$.

We will suppose that

(H(i)) $f_1(x, y_1, y_2)$ and $f_2(x, y_1, y_2)$ are continuous on S, and are hence bounded (because f_1 and f_2 are continuous functions on the closed bounded set S), say $|f_1(x, y)| + |f_2(x, y)| \le M$ on S.

(H(ii)) $f_1(x, y_1, y_2)$ and $f_2(x, y_1, y_2)$ are Lipschitz continuous with respect to (y_1, y_2) on S. That is, there exist L_1 and L_2 such that for $x \in [a - h, a + h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$
\begin{aligned}
|f_1(x, u_1, u_2) - f_1(x, v_1, v_2)| &\le L_1(|u_1 - v_1| + |u_2 - v_2|) \text{ and} \\
|f_2(x, u_1, u_2) - f_2(x, v_1, v_2)| &\le L_2(|u_1 - v_1| + |u_2 - v_2|).
\end{aligned}
$$

It is easy to see that these conditions are equivalent to the following:

(H(i))$'$ $\underline{f}(x, \underline{y})$ is continuous on S, and bounded by M, say (that is $||\underline{f}(x, \underline{y})||_1 \le M$). [$M$ must exist because $\underline{f}$ is a continuous function on the closed bounded set S.]

(H(ii))$'$ $\underline{f}(x, \underline{y})$ is Lipschitz continuous with respect to $\underline{y}$ on S. That is, there exists L such that for $x \in [a - h, a + h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$||\underline{f}(x, \underline{u}) - \underline{f}(x, \underline{v})||_1 \le L||u - v||_1.$$

Then,

Theorem 1.6. (Picard's existence theorem for systems.) *Suppose that conditions* **(H(i))** *and* **(H(ii))** *(or* **(H(i))**$'$ *and* **(H(ii))**$'$*) hold. Then there exists $0 < \eta \le h$, such that the initial value problem (1.31) has a unique solution for $x \in [a - \eta, a + \eta]$*

Our previous proof using the CMT will extend to this case if we work in the complete metric space $\mathcal{C}_\eta := \mathcal{C}([a - \eta, a + \eta]; B_k(\underline{b}))$, the space of continuous functions mapping from $[a - \eta, a + \eta]$ to $B_k(\underline{b})$ with norm (or distance) on $\mathcal{C}_\eta$ defined by

$$||\underline{y}||_{\sup} = \sup_{x \in [a-\eta, a+\eta]} ||\underline{y}(x)||_1 \left(:= \sup_{x \in [a-\eta, a+\eta]} (|y_1(x)| + |y_2(x)|). \right)$$

As before, we can write the initial value problem as an integral equation

$$\underline{y}(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

where by the integral we mean that we integrate componentwise. Provided $\underline{f}(s, \underline{y}(s))$ is continuous in s, $\underline{y}$ is a solution of the differential equation if and only if $\underline{y}$ is a solution of the integral equation.

If we define

$$(T\underline{y})(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

then we can write this as a fixed point problem

$$\underline{y} = T\underline{y}.$$

As before we can now work in the complete metric space $\mathcal{C}_\eta$, to show that, provided we choose $\eta < \min\{h, k/M, 1/L\}$, then $T : \mathcal{C}_\eta \to \mathcal{C}_\eta$ and is a contraction (see problem sheet).

Again we can extend the range of the solution to all $x \in [a - h, a + h]$, by iteration.

Corollary 1.7. *If $Mh \leq k$ then there is a unique solution for all $x \in [a - h, a + h]$*

Again if the functions are globally Lipschitz, then the solution is global.

This all extends easily to systems of n equations.

1.10   Picard for Higher Order ODEs

With Picard extended to first-order systems, it is a small step to extend it to a single, higher order ODE. For simplicity, we consider just an IVP for *linear* second-order ODEs (which will be considered in more detail in DEs2):

$$y'' + p(x)y' + q(x)y = r(x)$$

with initial data

$$y(a) = b \quad y'(a) = c,$$

and p, q, r continuous for $|x - a| \leq h$.

To reduce this to a first-order system, introduce $z = y'$ and write

$$y' = z := f_1(x, y, z)$$

$$z' = -pz - qy + r := f_2(x, y, z)$$

with data $y(a) = b, \quad z(a) = c$. This is precisely in the form to which the previous section applies, and it's easy to check that the global Lipschitz condition is satisfied, so we get:

Theorem 1.8. (Picard for second-order linear ODEs)

With the assumptions as above, the solution exists for $|x - a| \leq h$, and is unique.

Clearly this method can be extended to the IVP for an n-th order linear ODE. In particular, this justifies our belief that an n-th order ODE needs n pieces of data to fix a unique solution.

1.11 Summary

So we have looked at existence and uniqueness of solutions of various initial value problems. We found that Lipschitz continuity gives existence and uniqueness and that uniqueness can fail without the Lipschitz continuity. Even with Lipschitz continuity, the existence is often local, though a global Lipschitz condition on an interval containing the initial point will give existence and uniqueness of solutions on that interval.

There were two different methods of proof of existence and uniqueness. First we did a direct proof using successive approximations. Then we saw that using the CMT simplifies the proof, because the hard work has already been done in the proof of the CMT and proving completeness of $\mathcal{C}$. This proof readily extends to treat systems of ODEs (though here again we could have used successive approximation). A disadvantage of using the CMT was that it gave the result only for a restricted range of x, though it was easy to extend using iteration. This can be a feature of proofs using abstract results, because sometimes some of the detail is lost in the abstraction.

We also derived Gronwall's inequality, and used it to show that solutions depend continuously on the initial data. It also gave another proof of uniqueness.

2 Plane autonomous systems of ODEs

The definition: a *plane autonomous system of ODEs* is a pair of ODEs of the form;

$$\frac{dx}{dt} = X(x, y) \tag{2.1}$$
$$\frac{dy}{dt} = Y(x, y)$$

Here "autonomous" means there is no t-dependence in X or Y, and "plane" means there are just two equations, so we can draw pictures in the (x, y) - plane, which will then be called the *phase plane*. We will assume throughout that X and Y Lipschitz continuous in x and y. Then given initial values $x(0) = a$, $y(0) = b$, we expect to find a unique solution which will define *a trajectory* or *phase path* in the phase plane. It is convenient, though not necessary, to think of t as time, and the trajectory as tracing out the path in time of a moving particle. Then we can put an arrow on the trajectory giving the direction of increasing t. We will denote $\dot{x} = \frac{dx}{dt}$ etc.

2.1 Critical points and closed trajectories

A *critical point* is a point in the phase plane where $X = Y = 0$. So a critical point is a particular (very special) trajectory (for which the solution is constant so the particle doesn't move).

There is a trajectory through every point and trajectories can only cross at a critical point. Indeed, suppose that X and Y are Lipschitz continuous in x and y and that (x_0, y_0) is not a critical point, so, without loss of generality, we can assume that $X(x_0, y_0) \neq 0$, and hence, by continuity, that $X(x, y) \neq 0$ in a neighbourhood of the point. Thus $\frac{Y}{X}$ is Lipschitz continuous there. So

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{Y}{X}, \quad y(x_0) = y_0$$

has a unique solution and there will indeed be a unique path through the point. Thus there is a trajectory through every point and trajectories can only cross at a critical point. [1] □

There may be trajectories in the phase plane which are closed i.e. which return to the same point. Provided they don't contain a critical point, these correspond to **periodic solutions** of (2.1) as may be seen as follows:

Suppose the trajectory is closed so that for some finite value p of t, $(x(p), y(p)) = (x(0), y(0))$, while $(x(t), y(t)) \neq (x(0), y(0))$ for $0 < t < p$. Define $\bar{x}(t) = x(t + p)$,

---

[1] One might try to argue that if X and Y are Lipschitz continuous there is a unique (local) solution $x(t), y(t)$ such that $x(t_0) = x_0$ and $y(t_0) = y_0$. But this does not necessarily mean that there is only one trajectory through any point (x_0, y_0), as this point could also serve as an initial point for other starting times. Our proof shows this cannot happen.

$\bar{y}(t) = y(t + p)$. Then

$$\frac{d\bar{x}}{dt} = X(x(t + p),\ y(t + p)) = X(\bar{x}(t),\ \bar{y}(t))$$

and

$$\frac{d\bar{y}}{dt} = Y(\bar{x}(t),\ \bar{y}(t)).$$

So $(\bar{x}(t), \bar{y}(t))$ is another solution with $\bar{x}(0) = x(p) = x(0);\ \bar{y}(0) = y(p) = y(0)$. Now by uniqueness of solution (given Lipschitz again).

$$x(t + p) = \bar{x}(t) = x(t)$$

$$y(t + p) = \bar{y}(t) = y(t),$$

but this is now true for all t, so a **closed trajectory** corresponds to a **periodic solution** of (2.1) with period p. Converse is trivial. $\qquad\qquad\square$

2.1.1 An example

Consider the harmonic oscillator equation

$$\ddot{x} = -\omega^2 x.$$

Turn this into a plane autonomous system by introducing y as follows:

$$\left.\begin{array}{lll} \dot{x} = y & = X(x, y) \\ so \quad \dot{y} = -\omega^2 x & = Y(x, y). \end{array}\right\} \tag{2.2}$$

(Clearly this trick often works for second-order ODEs arising from Newton's equations.) The only critical point is $(0, 0)$, but note that

$$\frac{d}{dt}(\omega^2 x^2 + y^2) = 2\omega^2 x \dot{x} + 2y\dot{y} = 0$$

so $\omega^2 x^2 + y^2 = $ constant. (which, from Prelims dynamics, we know to be proportional to the total energy). For a given value of the constant this is the equation of an ellipse, so we can draw all the trajectories in the phase plane as a set of nested (concentric) ellipses:
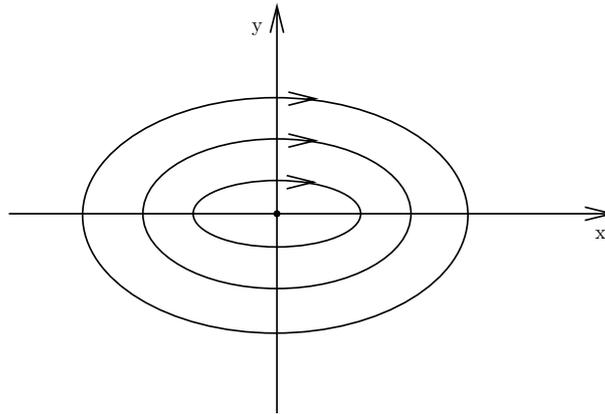
Figure 2.1: The phase diagram for the harmonic oscillator; to put the arrows on the trajectories, notice that $\dot{x} > 0$ if $y > 0$.

The picture in the phase plane is called the *phase diagram* (or *phase portrait*) and from that we see that all trajectories are closed, so all solutions are periodic (as we already know, from Prelims).

2.2 Stability and linearisation

We want to learn how to sketch the trajectories in the phase plane in general and to do this we first consider their *stability*. Intuitively we say a critical point (a, b) is *stable* if near (a, b) the trajectories have all their points close to (a, b) for all t greater than some t_0. We make the formal definition:

Definition A critical point (a, b) is *stable* if given $\epsilon > 0$ there exists $\delta > 0$ and t_0 such that for any solution $(x(t), y(t))$ of (2.1) for which $\sqrt{(x(t_0) - a)^2 + (y(t_0) - b)^2} < \delta$

$$\sqrt{(x(t) - a)^2 + (y(t) - b)^2} < \epsilon, \quad \forall t > t_0.$$

A critical point is *unstable* if it is not stable.

(Here we have used the Euclidean distance. We could use other norms such as l^1 or l^∞)

A common way to analyse the stability of a critical point is to linearise about the point and assume that the stability is the same as for the linearised equation. There are rigorous ways of showing when this is true. We will assume it is valid, pointing out the cases where it is likely fail. Linearising will also enable us to classify the critical points according to what the trajectories look like near the critical point .

So suppose $P = (a, b)$ is a critical point for (2.1), so

$$X(a, b) = 0 = Y(a, b). \tag{2.3}$$

Now $x = a$, $y = b$ is a solution of (2.1). We linearise by setting

$$x = a + \zeta(t); \ y = b + \eta(t)$$

27

where ζ and η are thought of as small. From (2.1), and Taylor's theorem

$$\dot{x} = \dot{\zeta} = X(a + \zeta, b + \eta) = X(a,b) + \zeta X_x|_p + \eta X_y|_p + \text{ h.o.}$$

$$\dot{y} = \dot{\eta} = Y(a,b) + \zeta Y_x|_p + \eta Y_y|_p + \text{ h.o.}$$

where 'h.o.' means quadratic and higher order terms in ζ and η. Now use (2.3) and neglect higher order terms to find

$$\left.\begin{aligned}
\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix}^{\cdot} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} X_x|_p & X_y|_p \\ Y_x|_p & Y_y|_p \end{pmatrix}
\end{aligned}\right\} \tag{2.4}$$

Call this (constant) matrix M and set $\underline{Z}(t) = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$ then (2.4) becomes

$$\dot{\underline{Z}} = M \underline{Z}. \tag{2.5}$$

We can solve (2.5) with eigen-vectors and eigen-values as follows: $\underline{Z}_0 e^{\lambda t}$ is a solution, with constant vector $\underline{Z}_0$ and constant scalar λ if

$$\lambda \underline{Z}_0 = M \underline{Z}_0,$$

i.e. $\underline{Z}_0$ is an eigen-vector of M with eigen-value λ. We are considering just 2×2-matrices, with eigen-values say λ_1 and λ_2 so the general solution if $\lambda_1 \neq \lambda_2$ is

$$\underline{Z}(t) = c_1 \underline{Z}_1 e^{\lambda_1 t} + c_2 \underline{Z}_2 e^{\lambda_2 t}, \tag{2.6}$$

for constants c_i. Recall λ_1, λ_2 may be real, in which case the c_i and the $\underline{Z}_i$ are real, or a complex conjugate pair, in which case the c_i and the $\underline{Z}_i$ are too.

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ say, we need to take more care. The Cayley-Hamilton Theorem (see Algebra I) implies that $(M - \lambda I)^2 = 0$ since the characteristic polynomial is $c_M(x) = (x - \lambda)^2$, so either $M - \lambda I = 0$ or $M - \lambda I \neq 0$. We have a dichotomy:

(i) if $M - \lambda I = 0$ then $M = \lambda I$ and the solution is

$$\underline{Z}(t) = \underline{C} e^{\lambda t} \tag{2.7}$$

for any constant vector $\underline{C}$.

(ii) if $M - \lambda I \neq 0$ then there exists a constant vector $\underline{Z}_1$ with

$$\underline{Z}_0 := (M - \lambda I)\underline{Z}_1 \neq 0$$

but

$$(M - \lambda I)\underline{Z}_0 = (M - \lambda I)^2 \underline{Z}_1 = 0.$$

(So $\underline{Z}_0$ is the one linearly independent eigenvector of M.) One now checks that the solution of (2.5) is

$$(c_1 \underline{Z}_1 + (c_0 + c_1 t)\underline{Z}_0)e^{\lambda t}. \tag{2.8}$$

Now we can use (2.6) and (2.8) to classify critical points.

2.3 Classification of critical points

We shall assume that neither eigenvalue of the matrix M is zero, which is the requirement that the critical point be *non-degenerate*. A proper discussion of this point would take us outside the course but roughly speaking if a critical point is *degenerate* then we need to keep more terms in the Taylor expansion leading to (2.4), and the problem is much harder.

Case 1. $0 < \lambda_1 < \lambda_2$ (both real of course)

From (2.6), as $t \to -\infty$, $\underline{Z}(t) \to 0$, and $\underline{Z}(t) \sim c_1 \underline{Z}_1 e^{\lambda_1 t}$ unless $c_1 = 0$ in which case $\underline{Z}(t) \sim c_2 \underline{Z}_2 e^{\lambda_2 t}$, while as $t \to +\infty$, $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_2$, unless $c_2 = 0$ when $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_1$
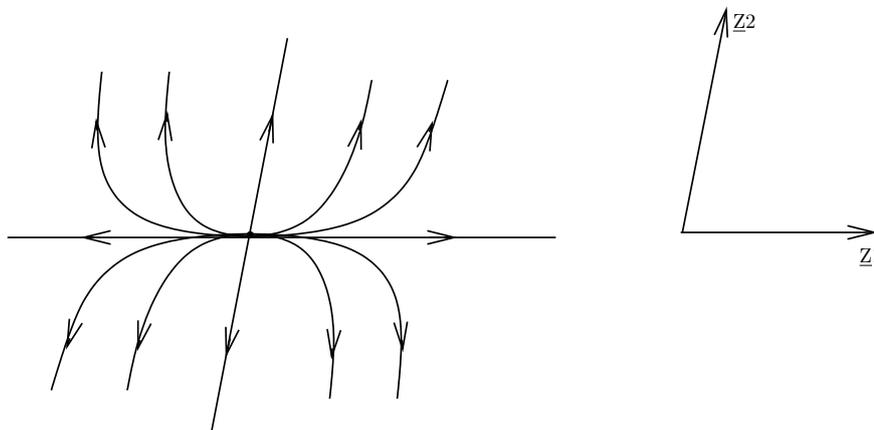
Figure 2.2: An unstable node.

These trajectories converge on the critical point into the past, but go off to infinity in the future. A critical point with these properties is called an **unstable node**.

Case 2: $\lambda_1 < \lambda_2 < 0$ (both real)

This is as above but with $t \to -t$ and the roles of $\underline{Z}_1$, $\underline{Z}_2$ switched. The trajectories converge on the critical point into the future and come in from infinity in the past.
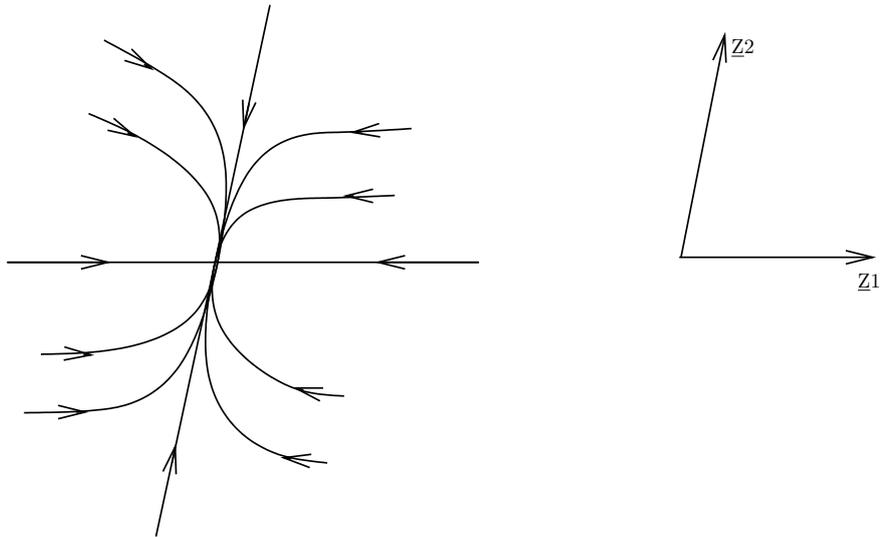
Figure 2.3: A stable node.

This is a **stable node**.

Case 3: $\lambda_1 = \lambda_2 = \lambda$. If the solution of the linearised equation is given by (2.7) (case (i)) we have a **star**, while if the solution is given by (2.8) (case (ii)) there is an **inflected node**. In both cases the critical point is stable if $\lambda < 0$ and unstable if $\lambda > 0$.
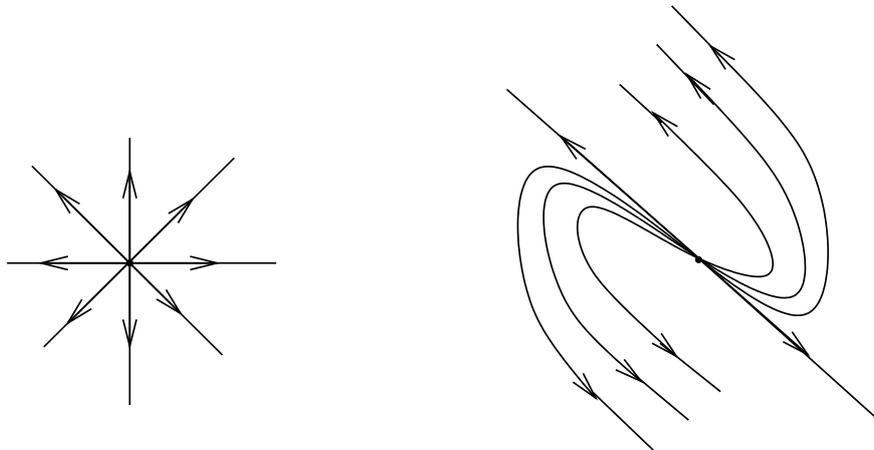
Figure 2.4: Unstable star case (i) and unstable inflected node case (ii)

Case 4: $\lambda_1 < 0 < \lambda_2$ (both real)

$$
\begin{array}{llll}
\text{If } c_1 = 0 \text{ then} & \underline{Z}(t) \to \infty & \text{along } \underline{Z}_2 & \text{as } t \to \infty \\
& \to 0 & \text{along } \underline{Z}_2 & \text{as } t \to -\infty. \\
\text{If } c_2 = 0 \text{ then} & \underline{Z}(t) \to 0 & \text{along } \underline{Z}_1 & \text{as } t \to \infty \\
& \to \infty & \text{along } \underline{Z}_1 & \text{as } t \to -\infty.
\end{array}
$$

30

If $c_1, c_2 \neq 0$ then $\underline{Z}(t) \rightarrow \infty$ along $\underline{Z}_2$ as $t \rightarrow \infty$ and along $\underline{Z}_1$ as $t \rightarrow -\infty$.

Most trajectories come in approximately parallel to $\pm\underline{Z}_1$ and go out becoming asymptotic to $\pm\underline{Z}_2$.
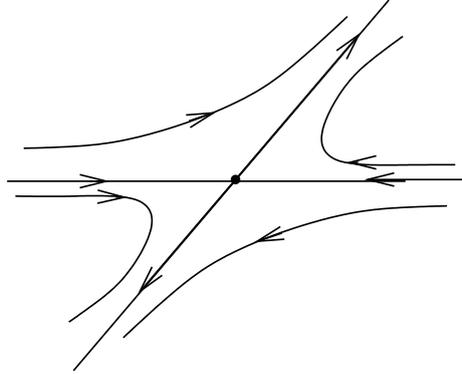
Figure 2.5: A saddle.

This is a **saddle** (to motivate the name, think of the trajectories as contour lines on a map; then two opposite directions from the critical point are uphill and the two orthogonal directions are downhill).

If the eigen-values are a complex conjugate pair we may write

$$\lambda_1 = \mu - i\nu, \quad \lambda_2 = \mu + i\nu \quad \mu, \nu \in \mathbb{R},$$

and the classification continues in terms of μ and ν.

In (2.6) the $c_i Z_i$ are a conjugate pair so if we put $c_1 = re^{i\theta}$, $\underline{Z}_1 = (1, ke^{i\phi})^T$, then

$$c_1 \underline{Z}_1 = \begin{pmatrix} re^{i\theta} \\ rke^{i(\phi+\theta)} \end{pmatrix}$$

so that

$$\underline{Z}(t) = e^{\mu t} \begin{pmatrix} 2r\cos(\nu t - \theta) \\ 2rk\cos(\nu t - (\phi + \theta)) \end{pmatrix}.$$

Case 5: $\mu = 0$

Then $\underline{Z}(t)$ is periodic.

This case is called a **centre**, and is stable. The sense of the trajectories, clockwise or anticlockwise, depends on the sign of B; $B > 0$ is clockwise (take $\zeta = 0$ and $\eta > 0$, then $\dot{\zeta} = B\eta > 0$).

31
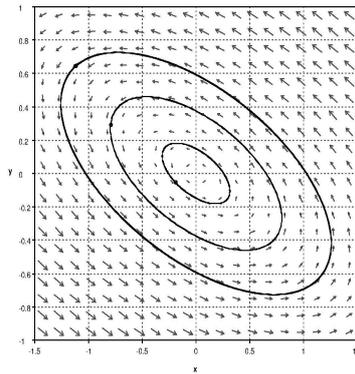
Figure 2.6: An anticlockwise centre $(B < 0)$; $X = -x - 3y$, $Y = x + y$

To see that this centre is stable: Take $t_0 = 0$. Consider the path whose maxmum distance from the critical point, (a, b), is $\epsilon > 0$. Let $\delta > 0$ be the minimum distance of this path from (a, b). Then
$$\sqrt{(x(0) - a)^2 + (y(0) - b)^2} \leq \delta \text{ implies } \sqrt{(x(t) - a)^2 + (y(t) - b)^2} \leq \epsilon, \text{ for all } t \geq 0.$$

Case 6: $\mu \neq 0$

This is just like case 5, but with the extra factor $e^{\mu t}$, which is monotonic in time. We have another dichotomy:

(i) $\mu > 0$ then $|\underline{Z}(t)| \to \infty$ as $t \to \infty$ so the trajectory spirals out, into the future. This is called an **unstable spiral**.
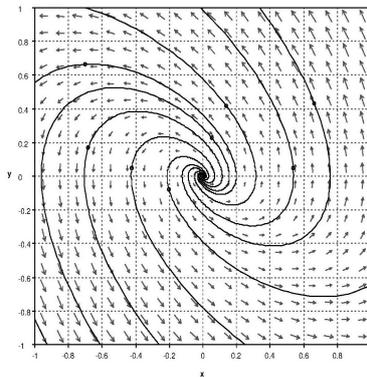
Figure 2.7: A antilockwise unstable spiral; X=-y, Y=x+y. Reverse the arrows for a stable spiral .

(ii) $\mu < 0$ this is the previous with time reversed so it spirals in, and is called a **stable spiral**.

32

In case 6, as in case 5, the sense of the spiral is dictated by the sign of B.

[An alternative method of looking at case 5 and 6:

Case 5: $\mu = 0$

so $\lambda_1 = -i\nu$ and $\lambda_i^2 = -\nu^2 < 0$; but, as both the trace and determinant of a matrix are invariant under $P^{-1}MP$ transformations, in terms of the matrix M of (2.4), trace M = $A + D = \lambda_1 + \lambda_2 = i\nu - i\nu = 0$ so det M = $AD - BC = -A^2 - BC = \lambda_1\lambda_2 = \nu^2 > 0$

Equation (2.4) becomes

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{2.9}$$

As an exercise, show that now $-C\zeta^2 + 2A\zeta\eta + B\eta^2$ is constant in time. We know that B, C have opposite signs with $(-BC) > A^2$ so this is the equation of an ellipse.

This case is called a **centre***.*

Case 6: $\mu \neq 0$

So, in (2.6), we must have $\underline{Z}_1 = \bar{\underline{Z}}_2$ and $c_1 = \bar{c}_2$ and

$$\underline{Z}(t) = e^{\mu t} \left[c_1 \underline{Z}_1 e^{-i\nu t} + \bar{c}_1 \bar{\underline{Z}}_1 e^{i\nu t} \right],$$

which is just like case 5, but with the extra factor $e^{\mu t}$, which is monotonic in time. So:

(i) *$\mu > 0$ then $|\underline{Z}(t)| \to \infty$ as $t \to \infty$ so the trajectory spirals out, into the future. An* **unstable spiral***.*

(ii) *$\mu < 0$ this is the previous with time reversed so it spirals in, a* **stable spiral***.]*

Important observation:

Both the trace and determinant of a matrix are invariant under $P^{-1}MP$ transformations, so in terms of the matrix M of (2.4), trace M = A + D = $\lambda_1 + \lambda_2$ and det M = AD − BC = $\lambda_1 \lambda_2$

Thus: if $A + D > 0$ then we have one of the cases 1, 4 or 6(i), all of which are **unstable** (but if $A + D < 0$ the critical point can be stable or unstable). Further det $M = \lambda_1 \lambda_2$. So when the eigenvalues are real the sign of det M tells us whether the signs of the eigenvalues are the same or different. The determinant is always positive in the case of complex eigenvalues.

Relationship to non-linear problem: One hopes that the lineariston will have the same type of critical point as the original system. In general if the linearisation has a node, saddle point or spiral, then so does the original system, but proving this is beyond the scope of this course. However, a centre in the linearisation does not imply a centre in the nonlinear system. This is not surprising when one reflects that a centre in the linear system arises when Re $\lambda = 0$ so the perturbation involved when one returns to the nonlinear system, however small, can change this property.

Analysing the critical points and their local behaviour is important in determining the general behaviour of trajectories of an ODE system. Connecting the various critical points together requires care. It helps to remember that trajectories can only cross at critical points. Also that the signs of X and Y give the signs of dx/dt and dy/dt respectively. One method to assist in putting the "time arrows" on the trajectories and sketching them is the use of "nullclines". A nullcline is a curve on which either $dx/dt = 0$ or $dy/dt = 0$. Such nullclines obviously cross at critical points. To find the nullclines sketch the curves $X(x, y) = 0$ and the curves $Y(x, y) = 0$. In particular in any region bounded by nullclines the trajectories must have a single sign for dx/dt and for dy/dt. Hence a simple examination of the expressions for X and Y in any region will determine if all the arrows in that region are "up and to the left", "up and to the right", "down and to the left" or "down and to the right".

2.3.1 An example

Find and classify the critical points for the system

$$\begin{aligned} \dot{x} &= x - y = X(x, y) \\ \dot{y} &= 1 - xy = Y(x, y) \end{aligned}$$

(2.10)

Solution: for the critical points, from $X = 0$ deduce $x = y$, therefore from $Y = 0$ deduce $x^2 = 1$, and we have either $(1, 1)$ or $(-1, -1)$.

For the classification, calculate

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix},$$

and evaluate at the critical points:

$$\text{at } (1, 1) : M = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} : \quad \lambda^2 - 2 = 0 : \quad \lambda = \pm\sqrt{2}$$

this is a **saddle**. The corresponding eigenvectors are:

$$\lambda_1 = -\sqrt{2} \quad \underline{Z}_1 = \begin{pmatrix} 1 \\ 1 + \sqrt{2} \end{pmatrix} \quad \text{direction in}$$

$$\lambda_2 = \sqrt{2} \quad \underline{Z}_2 = \begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix} \quad \text{direction out}$$

$$\text{at } (-1, 1) : \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \quad \lambda^2 - 2\lambda + 2 = 0 : \quad \lambda = 1 \pm i.$$

this is an **unstable spiral**; $B < 0$, so its described anticlockwise.
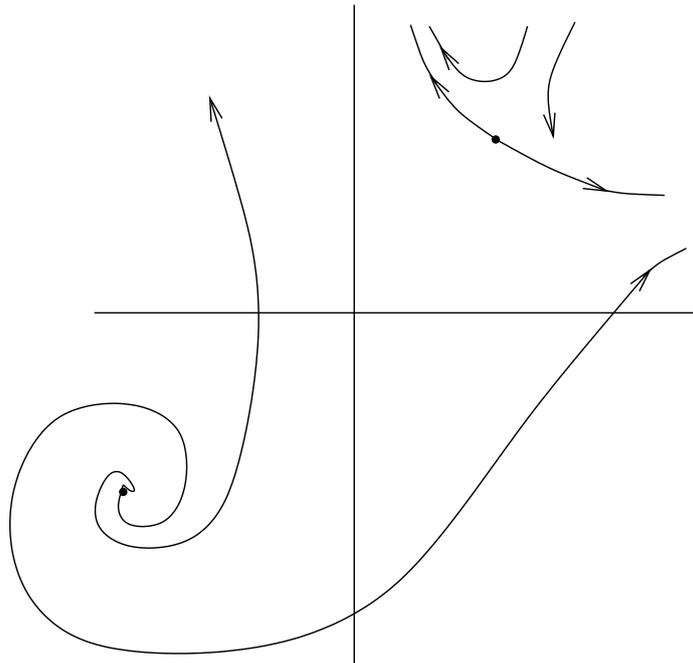
Figure 2.8: The phase diagram of (2.10)
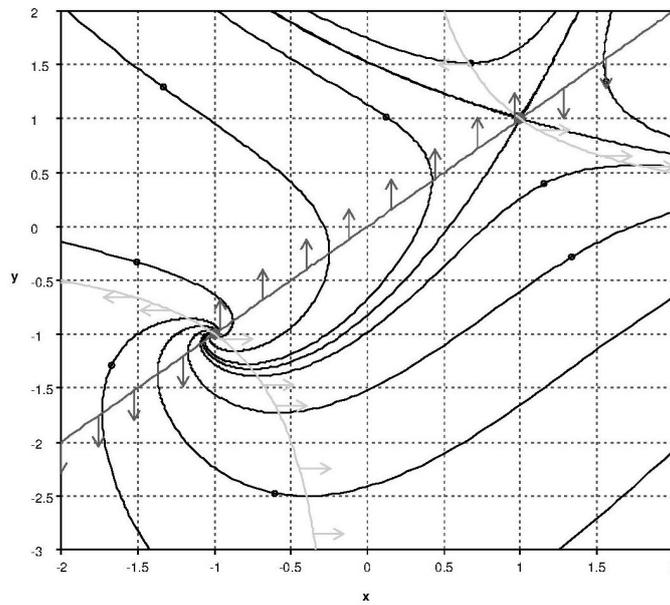
Figure 2.9: The phase plane diagram of (2.10) showing the nullclines $y = x$ and $xy = 1$.

2.3.2 Further example: the damped pendulum

Another example from mechanics: a simple plane pendulum with a damping force proportional to the angular velocity. We shall use the analysis of plane autonomous systems to understand the motion.

Take θ to be the angle with the downward vertical, then Newton's equation is

$$ml\ddot{\theta} = -mg\sin\theta - mkl\dot{\theta},$$

where m is the mass of the bob, l is the length of the string, g is the acceleration due to gravity and k is a (real, positive) constant determining the friction. We cast this as a plane autonomous system in the usual way: set $x = \theta$ and $y = \dot{x} = \dot{\theta}$ so

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l}\sin x - ky \end{aligned}$$

For simplicity below, we'll also assume that $k^2 < \frac{4g}{l}$, so that the damping isn't too large.

To sketch the phase diagram, we first find and classify the critical points. The critical points satisfy $y = 0 = \sin x$, so are located at $(x, y) = (N\pi, 0)$. Then

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x & -k \end{pmatrix}$$

The classification depends on whether N is even or odd:

$$\text{for } x = 2n\pi \quad M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -k \end{pmatrix}$$

which gives a stable spiral (clockwise);

$$\text{for } x = (2n+1)\pi \quad M = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -k \end{pmatrix}$$

which gives a saddle.

We now have enough information to sketch the phase diagram (note that $\dot{x}$ is positive or negative according as y is).
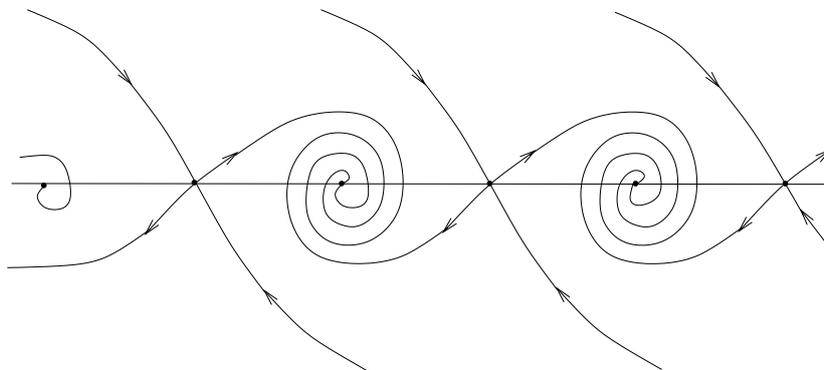
Figure 2.10: The phase diagram of the damped pendulum

2.3.3   An important example: The Lotka–Volterra predator-prey equations

This is a simplified mathematical model of a *predator-prey* system. Think of variables x standing for the population of prey, and y for the population of predators, both functions of t for time. As time passes, x increases as the prey breed, but decreases as the predators predate; likewise y increases by predation but decreases if too many predators compete. We assume that x and y are governed by the following plane autonomous system:

$$
\begin{aligned}
\dot{x} &= \alpha x - \gamma xy \\
\dot{y} &= -\beta y + \delta xy,
\end{aligned}
\tag{2.11}
$$

where α, β, γ, δ are positive real constants. Because of the interpretation as populations, we only care about $x \geq 0$, $y \geq 0$ but we shall consider the whole plane for simplicity. Again, the aim is to use the analysis of plane autonomous systems to lead us to the phase diagram and an understanding of the dynamics.

For the critical points first, set

$$X := x(\alpha - \gamma y) = 0$$

$$Y := y(-\beta + \delta x) = 0.$$

There are two solutions, $(0,0)$ and $(\frac{\beta}{\delta}, \frac{\alpha}{\gamma})$. For the matrix:

$$
M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \delta y & -\beta + \delta x \end{pmatrix}
$$

so first

$$
\text{at } (0,0): \ M = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}
$$

which gives a saddle, where, it is easy to see, the out-direction is the x-axis and the in-direction is the y-axis. Next

$$
\text{at } \left(\frac{\beta}{\delta}, \frac{\alpha}{\gamma} \right): \ M = \begin{pmatrix} 0 & -\frac{\beta \gamma}{\delta} \\ \frac{\alpha \delta}{\gamma} & 0 \end{pmatrix} : \lambda^2 + \alpha \beta = 0
$$

which gives a centre, described anticlockwise since $B < 0$.

We have found and classified the critical points. Before sketching the phase diagram, it is worth noting, from (2.11), that the axes are particular trajectories, and trajectories can only cross at critical points (as noted before).
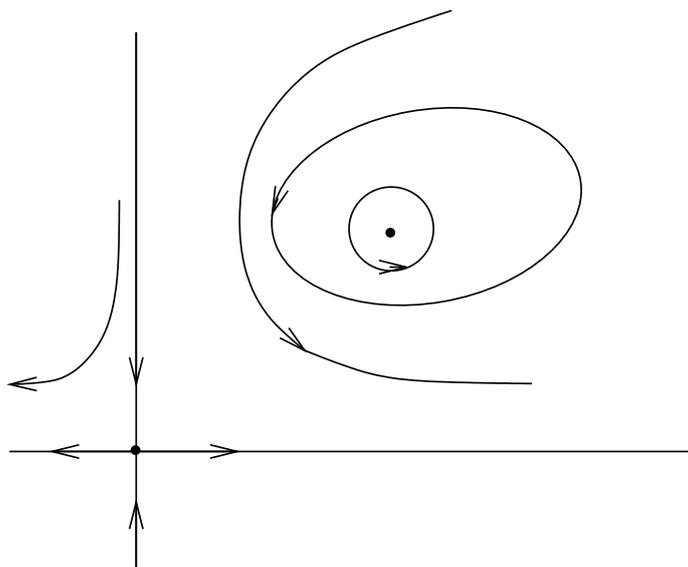
Figure 2.11: The phase diagram for the Lotka–Volterra system

Therefore any trajectory which is ever in the first quadrant is confined to the first quadrant, and no trajectory can enter the first quadrant from outside. Since there is a centre in the first quadrant, it looks as though all trajectories in the first quadrant may be **periodic**. This is true, and can be seen by the following argument: form the ratio

$$\frac{\dot{y}}{\dot{x}} = \frac{y(-\beta + \alpha x)}{x(\alpha - \gamma y)} = \frac{dy}{dx}.$$

and separate

$$\frac{(\alpha - \gamma y)}{y} dy - \frac{(-\beta + \delta x)}{x} dx = 0;$$

now integrate

$$\beta \log x - \delta x + \alpha \log y - \gamma y = C. \tag{2.12}$$

for a constant C. For different values of C, (2.12) is the equation of the trajectory or equivalently the trajectories are the level sets or contours of the function on the left in (2.12). This function is of the form $h(x) + k(y)$ and can easily be seen to have single maximum (at $(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma})$) . Also it tends to minus infinity on the axes and at infinity. Therefore its contours are all closed curves and so all the trajectories are closed and all the solutions of (2.11) are periodic.

This useful technique can be applied to other examples.

2.3.4   Another example from population dynamics.

This is a simple model for two species in competition. Suppose that, when suitably scaled, the population on an island of rabbits ($x \geq 0$) and sheep ($y \geq 0$) satisfies the

39

plane autonomous system:

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y). \qquad (2.13)$$

(The populations are in competition for resources so each has a negative effect on the other)

If we analyse this system we find that the critical points are $(0,0)$, $(3,0)$, $(0,2)$, $(1,1)$.

Then at $(0,0)$:

$$M = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

which has eigenvalues 3 and 2, with eigenvectors are (1,0), and (0,1) and is an unstable node.

At $(3,0)$:

$$M = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

which has eigenvalues -3 and -1, with eigenvectors (1,0), and (-3,1) and is a stable node.

At $(0,2)$:

$$M = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

which has eigenvalues -1 and -2, with eigenvectors (-1,2), and (0,1) and is a stable node.

At $(1,1)$:

$$M = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

which has eigenvalues $-1 - \sqrt{2}$ and $-1 + \sqrt{2}$, with eigenvectors $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$. and is a saddle point.

Again, as x and y represent populations we require that any trajectory which starts out in the first quadrant will remain there. As in the previous example this is indeed the case as the axes are particular trajectories.

Looking at the phase diagram we can see that, in the long term, depending on the initial data, either the rabbits or the sheep will survive.

Other values of the coefficients will give different outcomes - see problem sheet 2.

2.3.5 Another important example: limit cycles

Consider the plane autonomous system:

$$\dot{x} = (1 - (x^2 + y^2)^{\frac{1}{2}})x - y$$

$$\dot{y} = (1 - (x^2 + y^2)^{\frac{1}{2}})y + x.$$
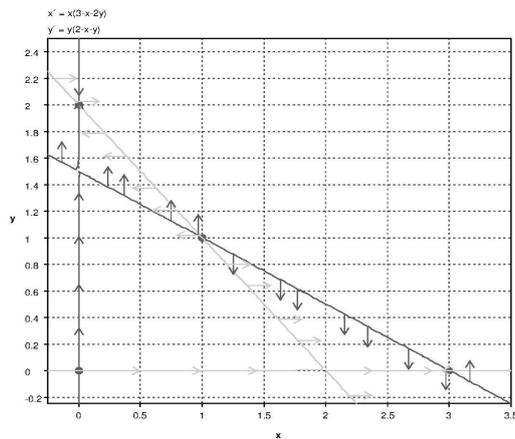
Figure 2.12: The nullclines for the equations (2.13).
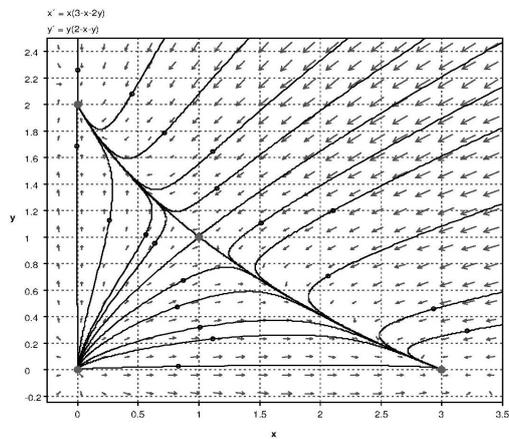
Figure 2.13: Phase diagram for the equations (2.13) for competitive species - no nullclines.
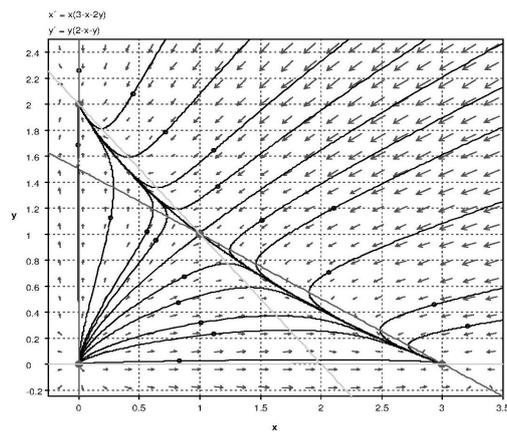
Figure 2.14: The phase diagram for the equations (2.13) for competitive species - with the nullclines Rabbits or sheep survive, depending on the initial data.

41

Put $x^2 + y^2 = r^2$ then

$$X = x(1 - r) - y$$
$$Y = y(1 - r) + x$$

and one sees that only critical point is $(0,0)$. One could go through the classification for this to find that it is an unstable spiral (exercise!).

Alternatively, in this case, we can analyse the full nonlinear system. We shall transform to polar coordinates. The simplest way to do this is as follows: first

$$r\dot{r} = x\dot{x} + y\dot{y} = x[x(1 - r) - y] + y[y(1 - r) + x]$$
$$= r^2(1 - r)$$

or

$$\dot{r} = r(1 - r).$$

Then, with

$$y = r \sin \theta,$$

we find

$$\dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta} = y(1 - r) + x,$$

which gives $\dot{\theta}$, so the system becomes

$$\left.\begin{array}{l} \dot{\theta} = 1 \\ \dot{r} = r(1 - r). \end{array}\right\}$$

Unlike the system in its previous form, we can solve this. First

$$\theta = t + \text{ const},$$

and then

$$\int dt = \int \frac{dr}{r(1 - r)} = \int \left(\frac{1}{r} + \frac{1}{1 - r} \right) dr$$

so

$$\log \frac{r}{|1 - r|} = t + \text{ const}$$

i.e.

$$\frac{r}{1 - r} = Ae^t.$$

Solve for r and change the constant:

$$r = \frac{1}{1 + Be^{-t}} = \frac{1}{1 + (\frac{1}{r_0} - 1)e^{-t}}$$

where $r(0) = r_0$.

Note that as $t \to \infty$, $r \to 1$, while as $t \to -\infty$ either $r \to 0$ if $r_0 < 1$ or $r \to \infty$ at some finite t if $r_0 > 1$.

Now it is clear that the origin is an unstable spiral, and that the trajectories spiral out of it anticlockwise. We can also see that $r = 1$ is a closed trajectory and that all other trajectories (except the fixed point at the origin) tend to it; we call such a closed trajectory a **limit cycle**. It is **stable** because the other trajectories converge on it. (For an example of an unstable limit cycle we could consider the same system but with t changed to $-t$.)
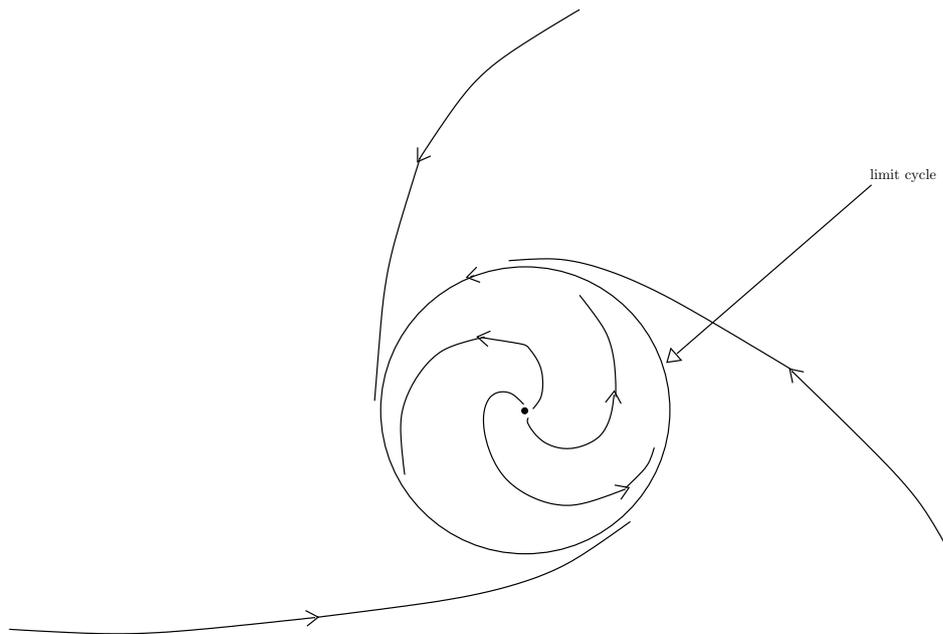
Figure 2.15: Phase diagram with a limit cycle

Another system with a limit cycle arises from the *Van der Pol equation*:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

where ϵ is a positive real constant. If $\epsilon = 0$ this is the harmonic oscillator again. If $\epsilon \neq 0$ then the usual trick produces a plane autonomous system:

$$\dot{x} = y$$

$$\dot{y} = -\epsilon(x^2 - 1)y - x.$$

The only critical point is $(0,0)$ and it's an unstable spiral for $\epsilon > 0$ (exercise!).

Claim: Its beyond us to show this, but this system has a unique limit cycle, which is stable. There are some good illustrations for this in e.g. Boyce and di Prima (pp 496–500 of the 5th edition).

2.4   The Bendixson–Dulac Theorem

It's important to be able to detect periodic solutions, but it can be tricky. We end this section with a discussion of a test that can rule them out.

Theorem 2.1. (Bendixson–Dulac)   *Consider the system $\dot{x} = X(x,y)$, $\dot{y} = Y(x,y)$, with $X, Y \in C^1$. If there exists a function $\varphi(x,y) \in C^1$ with*

$$\rho := \frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y) > 0$$

in a simply connected region R then there can be no nontrivial closed trajectories lying entirely in R.

Proof. (By nontrivial, I mean I want the trajectory must have an inside i.e. it isn't just a fixed point.) So suppose C is a closed trajectory lying entirely in R and let D be the disc (which also lies entirely in R, as R is simply connected) whose boundary is C. We apply Green's theorem in the plane. Consider the integral

$$\int\int_D \rho \; dxdy = \int\int_D \left[\frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y) \right] \; dxdy$$

$$= \oint_C -\varphi Y \; dx + \varphi X \; dy$$

$$= \oint_C -\varphi \left(-\dot{y}dx + \dot{x}dy \right).$$

But on C, $dx = \dot{x}dt$, $dy = \dot{y}dt$ so this is zero, which contradicts positivity of ρ, so there can be no such C.

2.4.1   Corollary.

If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

has fixed sign in a simply connected region R, then there are no nontrivial closed trajectories lying entirely in R.

This is just the previous but with φ const — in an example, always try this first!

2.4.2   Examples

(i) the damped pendulum (section 2.3.2)

$$\dot{x} = y$$

$$\dot{y} = -\frac{g}{l}\sin x - ky$$

has no periodic solutions.
Here

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -k \ < 0;$$

now use the corollary.

(ii)

$$\ddot{x} + f(x)\dot{x} + x = 0$$

has no periodic solutions in a simply connected region where f has a fixed sign.

By the usual trick we get the system

$$\dot{x} = y$$

$$\dot{y} = -yf(x) - x$$

then

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -f(x)$$

and we use the corollary.

(iii) The system

$$\dot{x} = y$$

$$\dot{y} = -x - y + x^2 + y^2$$

has no periodic solutions.

The corollary doesn't help so try the general case:

$$\rho := (\varphi X)_x + (\varphi Y)_y = \varphi(-1 + 2y) + X\varphi_x + Y\varphi_y.$$

Now guess: $\varphi_y = 0$ then

$$\rho = \varphi(-1 + 2y) + y\varphi_x = -\varphi + y(\varphi_x + 2\varphi)$$

so if we take $\varphi = -e^{-2x}$ the coefficient of y (which can take either sign) is zero and $\rho = 2e^{-2x} > 0$ and we are done.

45

PART II Partial Differential Equations.

3 First order semi-linear PDEs: the method of characteristics

3.1 The problem

In this chapter, we consider *first-order* PDEs of the following form:

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z) \tag{3.1}$$

The PDE is said to be *semi-linear* as it is linear in the highest order partial derivatives, with the coefficients of the highest order partial derivatives depending only on x and y. If P and Q depend also on z the PDE is said to be *quasi-linear*. We will consider only semi-linear equations.

We will assume throughout this section that, in the region specified, $P(x,y)$ and $Q(x,y)$ are Lipschitz continuous in x and y and $R(x,y,z)$ is continuous and Lipschitz continuous in z. This will be enough to ensure that the characteristic equations have a solution through each point in the region, which is unique except at points where $P = 0$ and $Q = 0$ (see below) .

We want to find a unique solution to (3.1) given suitable data and determine its *domain of definition*. This is the region in the (x,y)-plane in which the solution is uniquely determined by the data. It turns out to depend on both the equation and the data.

The solution of (3.1) will be a function

$$z = f(x,y)$$

but can be thought of as the surface defined by this equation, or equivalently defined by the equation

$$\Sigma(x,y,z) := z - f(x,y) = 0. \tag{3.2}$$

We shall refer to this as the *solution surface* and call it Σ. The method of solution of the equation will be to generate Σ.
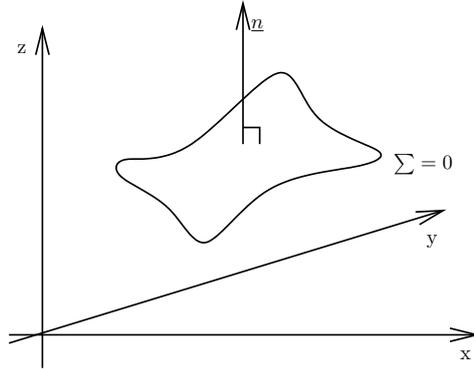
Figure 3.1: The solution surface

A normal to the solution surface is defined by

$$\mathbf{n} = \nabla\Sigma = \left(\frac{\partial\Sigma}{\partial x}, \frac{\partial\Sigma}{\partial y}, \frac{\partial\Sigma}{\partial z}\right) = (-f_x, -f_y, 1).$$

(this is a fact from Prelims for the single-subject mathematicians; Maths and Comp students should ask their tutors for enlightenment) so consider the vector $\mathbf{t} = (P, Q, R)$. Then

$$\mathbf{t} \cdot \mathbf{n} = -P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R,$$

which vanishes by (3.1), so $\mathbf{t}$ is tangent to the surface Σ.

3.2   The big idea: characteristics

We look for a curve Γ whose tangent is $\mathbf{t}$. If $\Gamma = (x(t), y(t), z(t))$ in terms of a parameter t this means

$$\left.\begin{aligned}
\frac{dx}{dt} &= P(x, y), & (a)\\
\frac{dy}{dt} &= Q(x, y), & (b)\\
\frac{dz}{dt} &= R(x, y, z). & (c)
\end{aligned}\right\} \qquad (3.3)$$

These are the *characteristic equations* and the curve Γ is a *characteristic curve* or just *a characteristic.* Given a characteristic $(x(t), y(t), z(t))$, call the curve $(x(t), y(t), 0)$, which lies below it in the (x, y)-plane, the *characteristic projection* or *characteristic trace.*

The next result shows that characteristics exist, and gives the crucial property of them:

Proposition 3.1. *Suppose that $P(x, y)$ and $Q(x, y)$ are Lipschitz continuous in x and y and $R(x, y, z)$ is continuous and Lipschitz continuous in z. Then*
(a) There is a characteristic projection through each point of the plane, and they can only meet at critical points (ie points where P and Q are both zero).
(b) Given a point $p \in \Sigma$, the characteristic through p lies entirely on Σ.

47

Proof

(a) This is exactly the same as in Section 2.1, as (3.3)(a),(b) is an autonomous system and, if $P \neq 0$, reduces to $\frac{dy}{dx} = \frac{Q}{P}$, which has a unique solution for given initial data. We can then find z uniquely along $y = y(x)$ from

$$\frac{dz}{dx} = \frac{R(x, y(x), z)}{P(x, y(x))}.$$

(b) (This is fiddly.) Let $(x(t), y(t), z(t))$ be the characteristic through p and set

$$\Sigma(t) = z(t) - f(x(t), y(t)).$$

Then to prove that the characteristic lies on the surface Σ we must show that $\Sigma(t) \equiv 0$. To do this we will show that $\Sigma(t)$ satisfies an IVP which has the unique solution zero so that $\Sigma(t) = 0$ for all t.

From (3.3) and the chain rule

$$\frac{d\Sigma}{dt} = R(x(t), y(t), z(t)) - \frac{\partial f}{\partial x} P(x(t), y(t))$$

$$- \frac{\partial f}{\partial y} Q(x(t), y(t)),$$

while from (3.1)

$$R(x(t), y(t), f(x(t), y(t))) - \frac{\partial f}{\partial x} P(x(t), y(t)) - \frac{\partial f}{\partial y} Q(x(t), y(t)) = 0.$$

Subtract these

$$\frac{d\Sigma}{dt} = R(x, y, z) - R(x, y, f)$$

Now put $f = z - \Sigma$, then the RHS here is a function $F(\Sigma(t), t)$ for some F with $F(0, t) = 0$. Furthermore, if the point p is $(x(0), y(0), z(0))$ then $\Sigma(0) = 0$.
Thus we have an ODE for $\Sigma(t)$, with given initial value, and the RHS is continuous and is Lipschitz continuous in Σ (check!), so the IVP has a unique solution. But $\Sigma(t) = 0$ is a solution, so it is the only one. Therefore, $\Sigma = 0$ all along Γ, and Γ lies on the solution surface. Q.E.D.

Thus the solution surface Σ is *generated* by a collection of characteristics.

3.2.1   Examples of characteristics

We need to gain proficiency in calculating characteristics.

(a) Calculate the characteristics for the PDE:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

From (3.3) write down the characteristic equations and solve them:

$$\frac{dx}{dt} = P = x; \quad x = Ae^t$$

$$\frac{dy}{dt} = Q = y; \quad y = Be^t$$

$$\frac{dz}{dt} = R = z; \quad z = Ce^t$$

with A, B, C constants (trivial to solve).

(b) Calculate the characteristics for the PDE:

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z.$$

$$\frac{dx}{dt} = y; \quad x = Bt + \frac{t^2}{2} + A$$

$$\frac{dy}{dt} = 1; \quad y = B + t$$

$$\frac{dz}{dt} = z; \quad z = Ce^t.$$

with A, B, C constants. To solve this system, pass over the first, solve the second, then come back to the first and third. (I am adopting a convention to introduce the constants A, B, C in the first, second and third of the characteristic equations respectively.)

In general solving the characteristic equations needs experience and luck; there isn't a general algorithm.

3.3 The Cauchy problem

A Cauchy Problem for a PDE is the combination of the PDE together with boundary data that, in principle, will give a unique solution, at least locally. We will look for suitable data and determine the domain on which the solution is uniquely determined.

Suppose we are given the solution z of (3.1) along a curve γ_0 (the *data curve*) in the (x, y)-plane. This produces a curve γ in space (the *initial curve*):
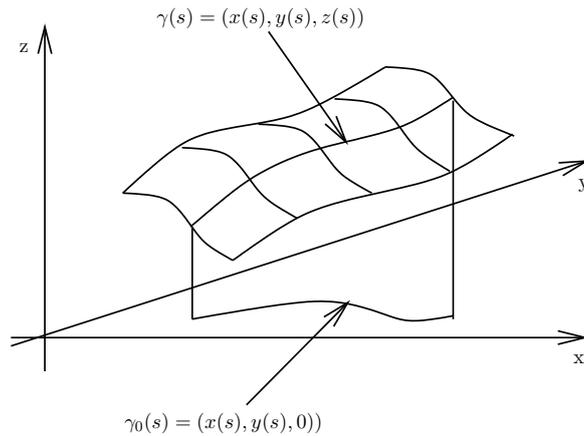
Figure 3.2: Geometry of the Cauchy problem.

We introduce a parameter s along γ so it is $(x(s), y(s), z(s))$, while γ_0 is the projection $(x(s), y(s), 0)$, and we will assume x, y, z are continuously differentiable. Then, to solve (3.1), we construct the solution surface Σ by taking the characteristics through the points of γ (because Proposition 3.1(b) tells us that the solution surface is generated by these characteristics) . Thus the method of solution, the *method of characteristics*, is

(i) Parametrise γ as $(x(s), y(s), z(s))$.

(ii) Solve

$$\frac{\partial x}{\partial t} = P$$

$$\frac{\partial y}{\partial t} = Q$$

$$\frac{\partial z}{\partial t} = R$$

for $(x(t, s), y(t, s), z(t, s))$ with data $x(0, s) = x(s)$; $y(0, s) = y(s)$; $z(0, s) = z(s)$.

then, knowing $(x(t, s), y(t, s), z(t, s))$, we have found Σ in parametric form. We would like to find the solution *explicitly*, that is z in terms of x and y, a question we will explore below, and there is a restriction on the data for the method to work, also to be found later.

3.4   Examples

(a) Solve

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z,$$

with $z(x, 0) = x$ for $1 \leq x \leq 2$.

50
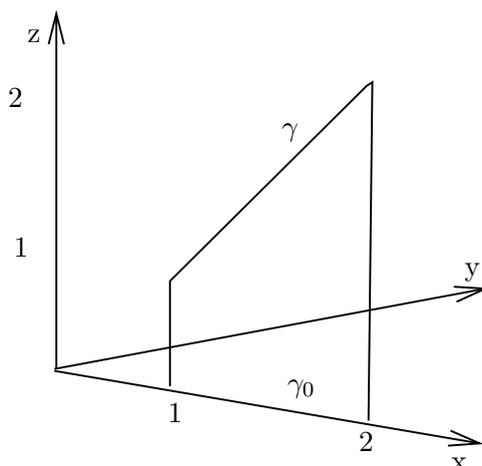
Figure 3.3: The data curve for this problem

We introduce a parameter s for the data, say $\gamma(s) = (s, 0, s)$, for $1 \le s \le 2$, and then solve the characteristic equations (done in section 3.2.1) with this as data at $t = 0$

$$x = Bt + \frac{t^2}{2} + A; \quad x(0, s) = A = s$$

$$y = B + t; \quad y(0, s) = B = 0$$

$$z = Ce^t; \quad z(0, s) = C = s$$

So, $C = s$, $B = 0$, $A = s$ and the parametric form of the solution is

$$\left. \begin{array}{l} x = s + \frac{1}{2}t^2 \\ y = t \\ z = se^t \end{array} \right\} \tag{3.4}$$

for $1 \le s \le 2$.

(b) (From Ockendon et al) Solve

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x + y)z \tag{3.5}$$

with $z = 1$ on the segment of the circle $(x - 2)^2 + y^2 = 2$, $y \ge 0$.

So we can take $\gamma(s) = (2 - \sqrt{2}\cos s, \sqrt{2}\sin s, 1)$, $s \in [0, \pi]$, and solve the characteristic equations:

$$\frac{\partial x}{\partial t} = P = x; \quad x = Ae^t; \quad A = (2 - \sqrt{2}\cos s)$$

$$\frac{\partial y}{\partial t} = Q = y; \quad y = Be^t; \quad B = \sqrt{2}\sin s$$

51

$$\frac{\partial z}{\partial t} = R = (x + y)z = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t)z.$$

We can integrate the final equation to get

$$\log|z| = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t) + C; \quad C = -(2 - \sqrt{2}\cos s + \sqrt{2}\sin s).$$

So the parametric form of the solution is

$$\left.\begin{array}{l} x = (2 - \sqrt{2}\cos s)e^t \\ y = (\sqrt{2}\sin s)e^t \\ z = \exp[((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)(e^t - 1)] \end{array}\right\} \tag{3.6}$$

3.5 Domain of definition

Where is the solution determined uniquely by the data? This is the domain of definition and is the region in the (x, y)-plane where the solution surface is uniquely determined and is given explicitly as $z = f(x, y)$.

Bounded Initial Data:

In general the initial curve could be bounded or semi-infinite. The solution surface is swept out by the characteristics through the initial curve, so the solution will be defined in the region swept out by the projections of the characteristics through the initial curve. In particular if the initial curve is bounded, the domain of definition will be bounded by the projections of the characteristics through the end points of the initial curve.

Blow up:

The method of characteristics reduces the PDE (3.1) to a system of ODEs. As we have already seen nonlinear ODEs can give rise to solutions which blow up, so the same must be true of non linear PDEs, even if those that are semi-linear.

Explicit solution: We want z as a function of x and y, so we need to be able to eliminate t and s in favour of x and y, at least in principle. For this, recall from Prelims the definition of the Jacobian:

$$J(s, t) = \frac{\partial(x, y)}{\partial(t, s)} = \det\begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}. \tag{3.7}$$

Now if

$$x = x(t, s), \text{ and } y = y(t, s)$$

are continuously differentiable functions of t and s in a neighbourhood of a point, then a sufficient condition to be able to find unique continuously differentiable functions

$$t = t(x, y) \text{ and } s = s(x, y)$$

in some neighbourhood of the point, is that J be non-zero at the point. We can then substitute into $z = z(t, s)$ to get

$$z = z(t(x, y), s(x, y)) = f(x, y),$$

a continuously differentiable function of x and y as required. This comes from a result known as the Inverse Function Theorem. We will have to take it on trust, but it is the two dimensional equivalent of the one dimensional result you saw in Analysis in Prelims - where the function f has a differentiable inverse if f has a non zero derivative.

Thus we require first that $J(s,0) \neq 0$ on the initial curve. If this is so, then the problem has a unique explicit solution, at least close to the initial curve. Then, we can continue to extend the solution away from the curve so long as $J \neq 0$ and the solution does not blow up.

This fails if the data curve touches a characteristic projection. For

$$J(s,0) = \det \begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix} = \det \begin{pmatrix} P & Q \\ x_s & y_s \end{pmatrix}. \tag{3.8}$$

So if, for some s, $J(s,0) = 0$ then, if a dash denotes differentiation with respect to s on $\gamma_0 = (x(s), y(s), 0)$, we have

$$P(x,y)y' - Q(x,y)x' = 0, \tag{3.9}$$

so the data curve is tangent to the projection of the characteristic at that point.

Typically away from the initial curve we will only have $J = 0$ if the characteristics cross each other (so that uniqueness fails) and for semi-linear problems this can only happen at critical points.

Examples:

Example (a) : Let us determine the domain of definition for the example of section 3.4(a):

The solution surface is swept out by the characteristics through γ, so has edges given by the characteristics through the ends of γ, which are at $s = 1$ and $s = 2$.

At $s = 1$, the characteristic projection is

$$x = 1 + \frac{1}{2}t^2; \quad y = t \quad \text{so} \quad x = 1 + \frac{1}{2}y^2;$$

at $s = 2$ it's

$$x = 2 + \frac{1}{2}t^2 \quad y = t \quad \text{so} \quad x = 2 + \frac{1}{2}y^2$$
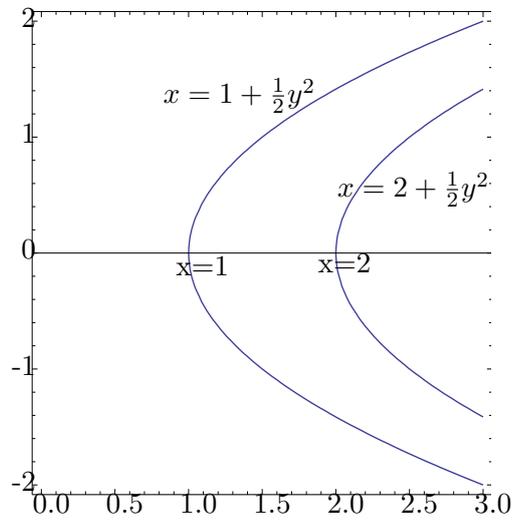
Figure 3.4: The domain of definition for this problem

so the solution surface lies above the region

$$1 + \frac{1}{2}y^2 \leq x \leq 2 + \frac{1}{2}y^2 \qquad (3.10)$$

and these curves don't meet .

It is easy to see from (3.4) that the solution can be given explicitly as $z(x,y) = (x - \frac{1}{2}y^2)e^y$ for $1 + \frac{1}{2}y^2 \leq x \leq 2 + \frac{1}{2}y^2$.

Thus the solution does not blow up and can be given explicitly. Therefore (3.10) gives the domain of definition. Or we could calculate J which is never zero.

Example (b): Now look at the example of section 3.4(b); First consider

$$J = \det \begin{pmatrix} (2 - \sqrt{2}\cos s)e^t & (\sqrt{2}\sin s)e^t \\ (\sqrt{2}\sin s)e^t & (\sqrt{2}\cos s)e^t \end{pmatrix}$$

$$= 2e^{2t}(1 - \sqrt{2}\cos s),$$

This vanishes when $s = \frac{\pi}{4}$, that is along the characteristic projection $y = x$, which touches the data curve at $(1,1)$. So we will restrict the data curve to $s \in [0, \pi/4)$. The data curve starts at $s = 0$, so the solution surface will have an edge given by the characteristic through the end of γ at $s = 0$ and at $s = 0$ the characteristic projection is $y = 0$. But the domain of definition is swept out by the projections of the characteristics through γ. So the domain of definition is $0 \leq y < x$. □

Remark: More generally if the data curve and characteristic projection coincide then, if the initial curve γ is a characteristic so that $z(s)$ satisfies $\frac{dz}{ds} = R$, there will be an infinity of solutions through γ, while otherwise there will be no solution.

For, if γ is a characteristic, then let C be any curve through γ whose projection is nowhere tangent to a characteristic projection. Then there is a solution surface through C. But this was any C so there is an infinity of solutions. On the other hand, if the data curve is a characteristic projection, then there can only be a solution if $\gamma = (x(s), y(s), z(s))$ is a characteristic, as if the solution surface is $z = f(x,y)$, then $z(s) = f((x(s), y(s))$ so, using the chain rule,

$$\frac{dz}{ds} = \frac{dx}{ds}f_x + \frac{dx}{ds} = Pf_x + Qf_yf_y = R.$$

So, returning to our example above, if one continues along the data curve from $s = \frac{\pi}{4}$ there will be problems as each characteristic meets the data curve in two points, so the initial data is likely to be inconsistent. If one changes the data curve to continue to the left of $y = x$ (for example the curve with the small dashes in the diagram) then it is likely there would only be problems on the characteristic $y = x$. But if we continue the data curve to include a section of the line $y = x$ (curve with larger dashes in the diagram) the problem is likely to have no solution.
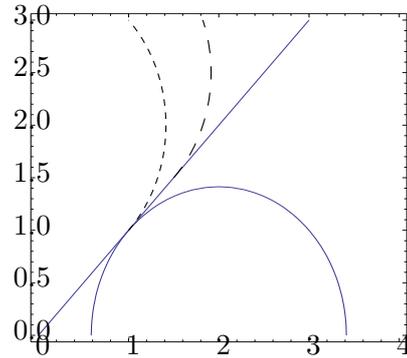
Figure 3.5: Different data curves for problem (3.5)

Example (c): Consider the PDE

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2,$$

with $z(x,0) = 1$.

parametrise the initial curve as $(s,0,1)$. Then the solutions of the characteristics are: $x = t + s$, $y = t$, $z = \frac{1}{1-t}$, so $z = \frac{1}{1-y}$ which blows up at $y = 1$.

3.6   Cauchy data and the Cauchy Problem:

We have seen that a necessary condition on the data for the solution to exist close to the initial curve is that on γ_0

$$P(x,y)y' - Q(x,y)x' \neq 0. \tag{3.11}$$

If it holds, we call the data *Cauchy data.*

Thus the Cauchy Problem is the PDE (3.1) with Cauchy data.

3.7   Discontinuities in the first derivatives

The characteristic projections have another property. They are the only curves across which the solution surface can have discontinuities in the first derivatives. For, suppose that γ is the curve $(x(s), y(s), z(s))$ in the solution surface, across which there are discontinuities in the first order partial derivatives but z is continuous. Use the superscript $\pm$ to denote the solution on either side of γ and denote the jumps in the partial derivative

by $[z_x]_-^+ = z_x^+ - z_x^-$ and $[z_y]_-^+ = z_y^+ - z_y^-$. Then differentiating each solution along γ, and also using (3.1)

$$\frac{dz^+}{ds} = z_x^+ \frac{dx}{ds} + z_y^+ \frac{dy}{ds}, \tag{3.12}$$

$$R = Pz_x^+ + Qz_y^+. \tag{3.13}$$

and

$$\frac{dz^-}{ds} = z_x^- \frac{dx}{ds} + z_y^- \frac{dy}{ds}, \tag{3.14}$$

$$R = Pz_x^- + Qz_y^-. \tag{3.15}$$

But z is continuous across γ, so $\frac{dz^+}{ds} = \frac{dz^-}{ds}$, and subtracting, we see that

$$0 = [z_x]_-^+ \frac{dx}{ds} + [z_y]_-^+ \frac{dy}{ds}, \tag{3.16}$$

$$0 = P[z_x]_-^+ + Q[z_y]_-^+. \tag{3.17}$$

Thus for there to be a non-zero jump, the solution of these simultaneous equations must be nonzero, so the determinant of the coefficients must be zero. That is

$$0 = \det \begin{pmatrix} P & Q \\ \frac{dx}{ds} & \frac{dy}{ds} \end{pmatrix}, \tag{3.18}$$

so that the projection of the curve must be a characteristic projection. That it is a characteristic curve now follows from (3.12) and (3.13). See problem sheet 3 for an example of this.
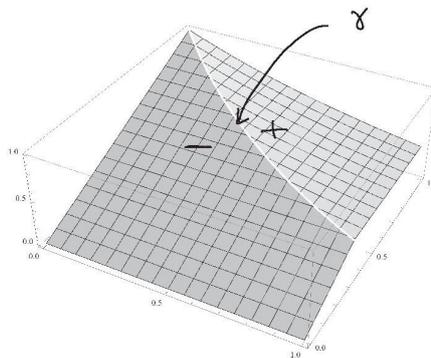
Figure 3.6: The curve γ.

3.8 General Solution

Another problem we could have considered, is what is the most general solution of (3.1)? Just as we expect the most general solution of an ODE to have n arbitrary constants, so we expect the most general solution of a PDE of order n to have n arbitrary functions.

For example: The first order PDE $\frac{\partial z}{\partial x}(x, y) = 0$, has the most general solution $z = f(y)$ where f is an arbitrary function.

4 Second order semi-linear PDEs

4.1 Classification

In this section, we are interested in second-order PDEs of the following form:

$$\underbrace{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}}_{\text{principal part}} = f(x,y,u,u_x,u_y). \qquad (4.1)$$

This PDE is said to be *linear* if f is linear in u, u_x, u_y, otherwise it is said to *semi-linear*. (If the coefficients a, b, c also depend on u, u_x, u_y it is said to be *quasi-linear*. We will consider only semi-linear equations.) You have seen the following examples in Prelims:

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace's equation}$$

$$u_{xx} - u_{yy} = 0 \quad \text{wave equation if } y = ct$$

$$u_{xx} - u_y = 0 \quad \text{heat equation if } y = t/\kappa.$$

Equations that are linear (in the dependent variable) have solutions that can be combined by linear superposition (taking linear combinations). In general PDEs that are nonlinear (for example where f, above, depends nonlinearly on u or its derivatives) do not have solutions that are superposable.

We will assume throughout that functions are suitably differentiable.

4.1.1 The idea:

In this section, the key idea is to change coordinates so as to simplify the principal part. So we make the change of variables

$$(x,y) \to (\varphi(x,y), \psi(x,y));$$

with non vanishing Jacobian (basically this ensures that the map is locally invertible):

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)} = \varphi_x \psi_y - \varphi_y \psi_x \neq 0.$$

We will abuse the notation a little and write (the solution) u as either a function of (x,y) or (φ,ψ) as required.

For the change in the partials, we calculate

$$u_x = u_\varphi \varphi_x + u_\psi \psi_x; \quad u_y = u_\varphi \varphi_y + u_\psi \psi_y$$

then

$$\left.\begin{array}{l} u_{xx} = u_{\varphi\varphi}\varphi_x^2 + 2u_{\varphi\psi}\varphi_x\psi_x + u_{\psi\psi}\psi_x^2 + u_\varphi\varphi_{xx} + u_\psi\psi_{xx} \\ u_{xy} = u_{\varphi\varphi}\varphi_x\varphi_y + u_{\varphi\psi}(\varphi_x\psi_y + \psi_x\varphi_y) + u_{\psi\psi}\psi_x\psi_y + u_\varphi\varphi_{xy} + u_\psi\psi_{xy} \\ u_{yy} = u_{\varphi\varphi}\varphi_y^2 + 2u_{\varphi\psi}\varphi_y\psi_y + u_{\psi\psi}\psi_y^2 + u_\varphi\varphi_{yy} + u_\psi\psi_{yy} \end{array}\right\} \qquad (4.2)$$

so that (4.1) becomes

$$A(\varphi,\psi)u_{\varphi\varphi} + 2B(\varphi,\psi)u_{\varphi\psi} + C(\varphi,\psi)u_{\psi\psi} = F(\varphi,\psi,u,u_\varphi u_\psi) \qquad (4.3)$$

with

$$\left.\begin{array}{l} A = a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 \\ B = a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y \\ C = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \end{array}\right\} \qquad (4.4)$$

(Beware, F will include lower order derivatives from (4.2).) In a matrix notation (4.4) is (check!)

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{pmatrix}$$

so that, taking determinants,

$$(AC - B^2) = (ac - b^2)(\varphi_x\psi_y - \psi_x\varphi_y)^2 = (ac - b^2)\left(\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right)^2. \qquad (4.5)$$

(We could obtain (4.5) directly from (4.4) but the matrix notation makes the computation simpler.) Now (4.5) leads to a classification of second-order linear PDEs:

4.1.2   The Classification

Second-order linear PDEs are classified into three types as follows:

1. $ac < b^2$ hyperbolic: e.g. wave equation;

2. $ac > b^2$ elliptic: e.g. Laplace equation;

3. $ac = b^2$ parabolic: e.g. heat equation.

So, by (4.5) the class of the equation is invariant under transformations with non-vanishing Jacobian.

We shall look at the classification in terms of the quadratic polynomial

$$a(x,y)\lambda^2 - 2b(x,y)\lambda + c(x,y) = 0. \qquad (4.6)$$

Note: We will assume that $a \neq 0$, in the domain under consideration. If $a = 0$ but $c \neq 0$, we can swap the roles of x and y.

Case 1: hyperbolic type

So $ac < b^2$ and the quadratic has distinct real roots λ_1, λ_2, say. So

$$a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\frac{dy}{dx} + c(x,y) = 0. \tag{4.7}$$

is equivalent to

$$\frac{dy}{dx} = \lambda_1(x,y), \quad \frac{dy}{dx} = \lambda_2(x,y). \tag{4.8}$$

Suppose these equations have solutions $\varphi(x,y)$=constant, $\psi(x,y)$ =constant, respectively. Set as change of variables

$$\varphi = \varphi(x,y), \quad \psi = \psi(x,y).$$

Then, on $\varphi(x,y)$=constant,

$$\varphi_x + \varphi_y\frac{dy}{dx} = 0$$

so that

$$\lambda_1\varphi_y = -\varphi_x$$

and thus

$$A(\varphi,\psi) = a(x,y)(\varphi_x)^2 + 2b(x,y)\varphi_x\varphi_y + c(x,y)(\varphi_y)^2 = 0,$$

and analogously $C(\varphi,\psi) = 0$. But $\lambda_1 \neq \lambda_2$, so $\frac{\varphi_x}{\varphi_y} \neq \frac{\psi_x}{\psi_y}$, and from (4.5) $B \neq 0$. Divide (4.3) by B to obtain the equation in the form

$$u_{\varphi\psi} = G(\varphi,\psi,u,u_\varphi,u_\psi). \tag{4.9}$$

This is the *normal form* (or *canonical form*) for a hyperbolic equation; the equation (4.7) is the *characteristic equation*; φ, ψ are *characteristic variables*; curves on which φ or ψ are constant are *characteristic curves*. We can often solve (4.9) explicitly.

Examples:

(a)
$$u_{xx} - u_{yy} = 0.$$

We already know how to solve this, but let us apply the method. So

$$a = 1, \; b = 0, \; c = -1, \text{and } \lambda^2 - 1 = 0.$$

We can take

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

and solve (4.8)

$$y'(x) = 1 \quad y'(x) = -1$$

61

to get
$$\varphi = x - y \quad \psi = x + y.$$

(There is clearly lots of choice at this stage.) The equation has become

$$u_{\varphi\psi} = 0,$$

which we solve at once by

$$u = f(\varphi) + g(\psi),$$

a solution known from Prelims. So the characteristic curves of the wave equation are $x + ct =$const and $x - ct =$const.

(b) An example with data: solve

$$xu_{xx} - (x + y)u_{xy} + yu_{yy} + \frac{(x + y)}{(y - x)}(u_x - u_y) = 0, \text{ for } y \neq x$$

with

$$u = \frac{1}{2}(x - 1)^2, \quad u_y = 0 \quad \text{on} \quad y = 1.$$

Problem is hyperbolic provided $x \neq y$ (check). The quadratic (4.6) is

$$x\lambda^2 + (x + y)\lambda + y = 0$$

$$= (\lambda + 1)(x\lambda + y);$$

so choose

$$\lambda_1 = -1 \quad \lambda_2 = -\frac{y}{x}$$

and solve

$$y'(x) = -1; \quad y'(x) = -\frac{y}{x},$$

by $x + y =$ const; $xy =$const, so put

$$\varphi = x + y; \quad \psi = xy.$$

Calculate

$$u_x = u_\varphi + yu_\psi$$
$$u_y = u_\varphi + xu_\psi$$

so that

$$u_{xx} = u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}$$
$$u_{xy} = u_{\varphi\varphi} + xu_{\varphi\psi} + yu_{\varphi\psi} + xyu_{\psi\psi} + u_\psi$$
$$u_{yy} = u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}.$$

(It is always better to calculate the derivatives directly, rather than trying to remember formulae.)

Now the PDE becomes

$$0 = x[u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}]$$
$$-(x+y)[u_{\varphi\varphi} + (x+y)u_{\varphi\psi}xyu_{\psi\psi} + u_\psi]$$
$$+y[u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}]$$
$$+(x+y)u_\psi$$
$$= (4xy - (x+y)^2)u_{\varphi\psi}$$

so

$$u_{\varphi\psi} = 0$$

and the solution is

$$u = f(\varphi) + g(\psi) = f(x+y) + g(xy).$$

To impose the data, calculate

$$u_y = f'(x+y) + xg'(xy)$$

so on $y = 1$,

$$u = f(x+1) + g(x) = \frac{1}{2}(x-1)^2$$
$$u_y = f'(x+1) + xg'(x) = 0.$$

Differentiate the first:

$$f'(x+1) + g'(x) = x - 1$$

and solve simultaneously with the second:

$$g'(x) = -1,$$

and integrate to find

$$g(x) = -x + c.$$

Substitute back in $u(x,1)$:

$$f(x+1) = \frac{1}{2}(x-1)^2 + x - c = \frac{1}{2}(x+1)^2 - x - c,$$

so

$$f(x) = \frac{1}{2}x^2 - x + 1 - c.$$

Finally

$$u = f(x+y) + g(xy) = \frac{1}{2}(x+y)^2 - (x+y) + 1 - xy.$$

Case 2: elliptic type

Now $ac > b^2$ so (4.6) has a complex conjugate pair of roots, and the integral curves of

$$y'(x) = \lambda(x, y); \quad y'(x) = \bar{\lambda}(x, y)$$

are in complex conjugate pairs, $\varphi(x, y)$ =const; $\psi(x, y) = \bar{\varphi}(x, y)$ =const. Then $A = C = 0$, $B \neq 0$ and the equation becomes

$$u_{\varphi\bar{\varphi}} = G(\varphi, \bar{\varphi}, u, u_\varphi, u_{\bar{\varphi}}).$$

Introduce new variables, ζ, η, given by $\varphi = \zeta + i\eta$, $\bar{\varphi} = \zeta - i\eta$, to obtain the normal form for an elliptic equation (check):

$$u_{\zeta\zeta} + u_{\eta\eta} = H(\zeta, \eta, u, u_\zeta, u_\eta), \tag{4.10}$$

which closely resembles Laplace's equation.

Example: Classify and reduce to normal form the PDE

$$y u_{xx} + u_{yy} = 0, \text{ for } y > 0.$$

$ac - b^2 = y$ so the equation is elliptic when $y > 0$.

The characteristic equation is
$$y(y')^2 + 1 = 0$$

that is
$$y^{1/2} y' = \pm i,$$

so integrating
$$2y^{3/2} \mp 3ix = \text{const.}$$

So take as variables $\zeta = 2y^{3/2}$; $\eta = 3x$. Making the substitution, we find that

$$3\zeta(u_{\zeta\zeta} + u_{\eta\eta}) + u_\eta = 0,$$

but $\zeta \neq 0$ so the normal form is

$$u_{\zeta\zeta} + u_{\eta\eta} = -\frac{u_\eta}{3\zeta}.$$

Case 3: parabolic type

Now $ac = b^2$ so (4.6) has a repeated root $\lambda(x, y)$. Solve $y'(x) = \lambda(x, y)$ for one new coordinate φ, and pick any ψ with

$$\varphi_x \psi_y - \varphi_y \psi_x \neq 0 \tag{4.11}$$

as the other, then $A = 0$ so $B^2 = AC = 0$. But $C \neq 0$, as $\psi =$const is not a characteristic curve by (4.11), so we get the normal form for a parabolic equation:

$$u_{\psi\psi} = G(u, \varphi, \psi, u_\varphi, u_\psi),$$

which closely resembles the heat equation.

Example:

Classify and reduce to normal form the equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0 \text{ for } x > 0. \tag{4.12}$$

The relevant quadratic is

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 = (x\lambda - y)^2$$

which has equal roots, so this equation is *parabolic*; $\lambda = \frac{y}{x}$ so solve

$$\frac{dy}{dx} = \frac{y}{x}, \text{ to get, for example, } \varphi = \frac{y}{x}$$

and take, for example, $\psi = x$. Calculate

$$u_x = -\frac{y}{x^2}u_\varphi + u_\psi$$

$$u_y = \frac{1}{x}u_\varphi$$

so that

$$u_{xx} = \frac{y^2}{x^4}u_{\varphi\varphi} - 2\frac{y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + 2\frac{y}{x^3}u_\varphi$$

$$u_{xy} = -\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi$$

$$u_{yy} = \frac{1}{x^2}u_{\varphi\varphi}.$$

The equation becomes

$$x^2\left[\frac{y^2}{x^4}u_{\varphi\varphi} + \frac{2y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + \frac{2y}{x^3}u_\varphi\right] +$$

$$+ 2xy\left[-\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi\right] + y^2\left[\frac{1}{x^2}u_{\varphi\varphi}\right] = x^2 u_{\psi\psi} = 0 \tag{4.13}$$

so the normal form is

$$u_{\psi\psi} = 0$$

with general solution $u = F(\varphi) + \psi G(\varphi)$. In terms of the original variables this is:

$$u = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right). \tag{4.14}$$

65

NB Very often, a question like this will be phrased in the form 'Classify and reduce to normal form the equation (4.12) and show that the general solution can be written as (4.14)'. Therefore candidates for φ and ψ are proposed by the question itself.

A warning example:

The type can change e.g. classify the equation

$$u_{xx} + y u_{yy} = 0.$$

Then

$$\lambda^2 + y = 0, \quad \lambda^2 = -y,$$

and this is:

- elliptic in $y > 0$,

- parabolic at $y = 0$,

- hyperbolic in $y < 0$.

4.2   Characteristics:

The characteristics of second order semi-linear PDEs have analogous properties to the characteristic *projections* of first order semi-linear PDEs. (Note the difference in terminology.)

Firstly, if there are discontinuities in the second derivatives of a solution across a given curve, then that curve must be a characteristic curve. To see this, suppose that the curve Γ, given parametrically by $(x(s), y(s))$, is a curve across which there are discontinuities in the second derivatives of the solution. Let u_{xx}^{+} etc denote values on one side of Γ and u_{xx}^{-} denote values on the other side of Γ. Then differentiating $u_x(x(s), y(s))$ and $u_y(x(s), y(s))$ along Γ, and noting that u, u_x, u_y are continuous across the curve,

$$\frac{du_x}{ds} = \frac{dx}{ds} u_{xx}^{\pm} + \frac{dy}{ds} u_{xy}^{\pm}$$
$$\frac{du_y}{ds} = \frac{dx}{ds} u_{yx}^{\pm} + \frac{dy}{ds} u_{yy}^{\pm}$$
$$\text{and also } f(x, y, u, u_x, u_y) = a(x, y) u_{xx}^{\pm} + 2b(x, y) u_{xy}^{\pm} + c(x, y) u_{yy}^{\pm}.$$

Subtracting the 'minus' equation from the 'plus' equation

$$0 = \frac{dx}{ds} [u_{xx}]_{-}^{+} + \frac{dy}{ds} [u_{xy}]_{-}^{+}$$
$$0 = \frac{dx}{ds} [u_{yx}]_{-}^{+} + \frac{dy}{ds} [u_{yy}]_{-}^{+}$$
$$0 = a(x, y) [u_{xx}]_{-}^{+} + 2b(x, y) [u_{xy}]_{-}^{+} + c(x, y) [u_{yy}]_{-}^{+},$$

where $[u_{xx}]^+_- = u^+_{xx} - u^-_{xx}$ denotes the jump in u_{xx} across Γ, etc. If there are to be discontinuities in the second derivatives, then this set of equations in the jumps must have a nonzero solution, so that the determinant of the coefficients must be zero. Thus

$$a(x,y)\left(\frac{dy}{ds}\right)^2 - 2b(x,y)\frac{dx}{ds}\frac{dy}{ds} + c(x,y)\left(\frac{dx}{ds}\right)^2 = 0, \qquad (4.15)$$

so Γ is a characteristic.

Furthermore, under suitable smoothness conditions, the Cauchy problem for a second order semi-linear PDE, where u and $\frac{\partial u}{\partial n}$ are given along a curve Γ, will have a unique local solution provided Γ is nowhere tangent to a characteristic curve. This result is beyond the scope of this course and will be investigated further in the Part B course, Applied PDEs. It can be seen that it is necessary that Γ is not a characteristic curve, as if u exists then it must have unique second order partial derivatives along Γ and exactly as above this can only be true when Γ is not a characteristic curve.

Remark: Our previous work carried the implicit assumption that $a \neq 0$. Note that (4.15) gives a method of calculating the characteristic curves if $a = 0$. In particular if $a = 0$ and $c = 0$ then the characteristic curves are $x =$const, $y =$const.

4.3 Type and data: well posed problems

We want to say something about the notion of *well posedness* and its connection with type. Our examples are mostly based on knowledge acquired in Prelims.

A problem, consisting of a PDE with data, is said to be *well posed* if the solution:

- exists

- is unique

- depends *continuously* on the data.

Recall that, in Section 1.7, we said that a solution of a DE is *continuously dependent on the data* if the error in the solution is small provided the error in the initial data is small enough.. We then gave a precise definition for ODEs. We now want to extend this definition to PDEs. Data can be given in different ways, so to be precise we will consider a problem where $u(x,y)$ is the solution of a certain PDE in a bounded subset of the plane D, with u given on some curve Γ. Then we will say that the solution depends continuously on the data if :

$\forall \epsilon > 0 \ \exists \delta > 0$ such that if u_i, $i = 1, 2$, are solutions with $u_i = f_i$ on Γ then

$$\sup_\Gamma |f_1 - f_2| < \delta \quad \Rightarrow \quad \sup_D |u_1 - u_2| < \epsilon.$$

The definition extends in a fairly obvious way to other types of data. (Note that there are plenty of other 'distances ' we could use in place of taking the sup, but that is what we will use here.)

Of the three requirements for well posedness it is existence which is the hardest to obtain. In Prelims solutions were found for a number of linear problems, either by making a change of variables and then integrating, or by using separation of variables and Fourier series. Anything more than this is beyond the scope of this course. Uniqueness of solution was also proved for a number of linear problems (even when you didn't know if the solution existed). Proving uniqueness and continuous dependence on the data is much easier for linear problems as we can then start out by looking at the difference between two solutions, which will then be the solution of some suitable problem. Later we will look at the linear equations Poisson's equation and the heat equation and state and prove the maximum principle, which will enable us to prove uniqueness and continuous dependence for suitable boundary data.

But first we need to consider what data might be appropriate. In Prelims you considered three particular PDEs, each with a different type of data which arose from the particular physical problem they modelled. These are summarised in the table below. It turns out that there are mathematical as well as physical reasons why each problem had a different type of data. So first we will look at some of these problems and consider which may be well posed.

Some Models from Prelims:

	PDE	Models	Boundary conditions
Wave Equation: Hyperbolic	$c^2 u_{xx} - u_{tt} = 0$	Waves on a string; u gives displacement	$u, \frac{\partial u}{\partial t}$ given $t = 0$ (IBVP) (Infinite string); $\frac{\partial u}{\partial t}$ given $t = 0$; plus end point condition - say $u = 0$ at ends (IVP) (Finite string)
Laplace's Equation: Elliptic	$u_{xx} + u_{yy} = 0,$ $(x,y) \in D$	Potential Theory or Steady Heat; u gives potential or temperature	u or $\frac{\partial u}{\partial n}$ given on boundary of D (BVP)
Heat Equation: Parabolic	$u_t = \kappa u_{xx}$	Heat; u gives temperature	u given at $t = 0$; plus endpoint conditions - u or $\frac{\partial u}{\partial x}$ given at ends (IBVP) (Temperature in finite bar)

Some examples from Prelims: (We will assume that the data is smooth enough for the following to hold.)

(a) Hyperbolic equation: The IVP and IBVP (initial-boundary-value problem) for the wave equation

$$u_{xx} - u_{yy} = 0 \quad (ct = y).$$

For the IVP (modelling an infinite string, where u is the displacement), we know the solution is

$$u = \frac{1}{2}\left[f(x+y) + f(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} g(s)ds, \tag{4.16}$$

where the data are $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$, $-\infty < x < \infty$. This is d'Alembert's solution of the IVP: it exists, and is unique and, intuitively at least, a small change in φ, ψ gives a small change in u (see problem sheet for proof). So this problem is well-posed.

For the IBVP (modelling a finite string length L, fixed at each end) consider the data:

$$u(x,0) = f(x), \quad u_y(x,0) = g(x) \quad 0 < x < L$$

$$u(0,y) = 0 = u(L,y).$$

So the boundaries are at $x = 0, L$. This IBVP is solved using separation of variables and Fourier series to get a solution

$$u = \sum_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right)$$

with a_n, b_n given in terms of f and g. Uniqueness was shown in Prelims, so this is the unique solution. If we now appeal to intuition for continuous dependence on the data this problem is well-posed.

(b) Elliptic equation:The BVP for the Laplace/Poisson's equation (modelling steady state heat, for example, where u is the temperature)

$$u_{xx} + u_{yy} = 0.$$

Do this first with data at the sides of a square, so $0 \leq x, y \leq a$ with

$$u(0,y) = u(a,y) = u(x,0) = 0; \; u(x,a) = f(x).$$

Consider separable solutions $u_n = \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$, then

$$u = \sum_n a_n \frac{\sinh\frac{n\pi y}{a}}{\sinh(n\pi)}\sin\frac{n\pi x}{a}$$

and

$$f(x) = \sum a_n \, \sin \frac{n\pi x}{a}$$

which determines the solution as a Fourier series.

Now a different BVP, with data at the circumference of the unit circle:

$$\text{on } r = 1, \ u = f(\theta)$$

and in polars

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

The separable solutions are

$$\begin{cases} \left(Ar^n + \frac{B}{r^n}\right)(C\cos n\theta + D\sin n\theta) \\ A + B\log r, \quad n = 0 \end{cases}$$

Regularity at $r = 0$ implies

$$u = \frac{1}{2}a_0 + \sum_1^\infty r^n(a_n \cos n\theta + b_n \sin n\theta)$$

and the boundary value at $r = 1$ requires

$$\frac{1}{2}a_0 + \sum(a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$$

which again is solved by Fourier methods.

Uniqueness was proved in Prelims so in each of these cases we have the unique solution. It is plausible, but beyond our scope, to show that there is existence of solution in general. Later we will prove that there is continuous dependence on the data. So this problem is well posed

(c) Parabolic equation: The IBVP for the heat equation (modelling heat flow in a bar length L, with the ends held at zero temperature, u is temperature.)

$$u_{xx} = u_y$$

on the semi-infinite strip where $y = t > 0$ and $0 < x < L$, and data

$$u(x, 0) = f(x); \ u(0, y) = 0 = u(L, y).$$

The relevant separable solutions are

$$u_n = \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2 t}{L^2}}$$

so that

$$u = \sum_n a_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2}t}$$

and the initial value requires

$$f(x) = \sum a_n \sin \frac{n\pi x}{L}$$

which is solved by Fourier methods. The solution exists *provided the series for u converges* which it will do for positive t. However, note that for negative t the exponentials grow rapidly with n and there is no reason to expect existence.

Uniqueness for this problem was done in Prelims, so again this is the unique solution for positive t. Later we will prove continuous dependence on the data. Thus the problem is well posed *forward in time.*

(d) What then is not well-posed? We give a few examples:

- BVPs for hyperbolic

  e.g. $u_{xx} - u_{yy} = 0$ on the unit square with data

  $$u(0, y) = u(1, y) = u(x, 0) = 0;\ u(x, 1) = f(x).$$

  Recall this data gave a well-posed problem for the Laplace equation, but here if $f = 0$, then $\sin n\pi x \sin n\pi y$ will do, for any n, while it can be proved that there is no solution at all if $f \neq 0$ (try the Fourier series to see what goes wrong).

- IVPs for elliptic

  e.g. $u_{xx} + u_{yy} = 0$ in the horizontal strip $0 \leq y \leq Y$, $-\infty < x < \infty$, with data

  $$u(x, 0) = 0,\ u_y(x, 0) = f(x).$$

  We know (from the problem sheet) that this data gives continuous dependence on the data for the wave equation. But not for Laplace's equation. For if $f(x) = \frac{1}{n} \sin nx$ it can be seen that $u(x, y) = \frac{1}{n^2} \sinh ny \sin nx$. But $\sup |\frac{1}{n^2} \sinh ny \sin nx| \to \infty$ as $n \to \infty$, whereas $1/n \sin nx \to 0$. Thus small changes in the initial data can lead to large changes in the solution. [More precisely: Suppose that there is continuous dependence on the initial data about the zero solution. That is: $\forall \epsilon > 0$, $\exists \delta > 0$ such that

  $$\sup_{x \in \mathbb{R}} |f(x) - 0| < \delta \Rightarrow \sup_{x \in \mathbb{R}, 0 \leq y \leq Y} |u(x, y) - 0| < \epsilon.$$

  But taking $\epsilon = 1$, say, there exists N such that for all $n > N$, $\sup |\frac{1}{n^2} \sinh ny \sin nx| > 1$, but for any $\delta > 0$ we can choose $n > N$ such that $|\frac{1}{n} \sin nx| < \delta$, giving a contradiction.

- IBVP for elliptic

  e.g. $u_{xx} + u_{yy} = 0$ on the semi-infinite strip $0 \leq x \leq 1$, $y \geq 0$, with data

  $$u(0, y) = u(1, y) = 0,\ u(x, 0) = 1,\ u_y(x, 0) = 0.$$

This data gives a well-posed problem for the wave equation. If we try for separable solutions here, we have $u_n = \sin n\pi x \cosh n\pi y$ so

$$u = \sum_n a_n \sin n\pi x \cosh n\pi y.$$

Initial conditions need

$$1 = \sum a_n \sin n\pi x$$

whence

$$a_n = 0 \quad n \text{ even}$$

$$= \frac{4}{n\pi} \quad n \text{ odd},$$

and then

$$u\left(\frac{1}{2}, y\right) = \sum_n \frac{4}{(2n+1)\pi}(-1)^n \cosh(2n+1)\pi y,$$

which does not converge for any $y > 0$ (because the "cosh" terms grow rapidly with n) - there is no solution (strictly speaking, we've only shown that there is no solution of the form considered; we need more).

- The BVP for the heat equation is not well-posed, but we won't show that.

Again, it is beyond our scope to prove it in this course, but these different behaviours are universal for the different types of second-order, linear PDEs. In tabulated form, which problems are well-posed?

	IVP	IBVP	BVP
Hyperbolic	yes	yes	no
Elliptic	no	no	yes
Parabolic	yes	yes	no

where the 'yes' for parabolic equations are only valid forward in time.

4.4   The Maximum Principle

4.4.1   Poisson's equation

The normal form for second-order elliptic PDEs is

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \tag{4.17}$$

73

The operator on the left-hand side is referred to as the Laplacian, for which the symbols $\nabla^2 u$ and Δu are often used as shorthand. Poisson's equation is a special case of (4.17), in which f depends only on x and y. We have already seen that appropriate boundary data for (4.17) is to give just one boundary condition on u everywhere on a closed curve.

We will consider the Dirichlet problem where u is given on the boundary of D:

$$
\begin{aligned}
u_{xx} + u_{yy} &= f(x,y) \quad \text{in } D & (4.18) \\
u &= g(x,y) \quad \text{on } \partial D. & (4.19)
\end{aligned}
$$

It was shown in Prelims, using the divergence theorem, that if a solution exists, then it is unique. Using the maximum principle we will give another proof of this and also show that there is continuous dependence on the data. The solution does exist, but apart from the particular cases considered in Prelims that is beyond the scope of this course.

Theorem 4.1. *(The Maximum principle for Poisson's equation) Suppose u satisfies Poisson's equation everywhere within a bounded domain, D,*

$$u_{xx} + u_{yy} = f(x,y) \quad (x,y) \in D, \tag{4.20}$$

where we have $f \geq 0$ in D. Then u attains its maximum value on ∂D.

Proof:

If we denote the boundary of D by ∂D , then as $D \cup \partial D$ is a closed bounded set, u must attain its maximum somewhere in D or on its boundary. The proof now proceeds in two parts.

Suppose first that $f > 0$ in D.

If u has an interior maximum at some point (x_0, y_0) inside D, then the following conditions must be satisfied at (x_0, y_0):

$$u_x = u_y = 0, \quad u_{xx} \leq 0, \quad u_{yy} \leq 0.$$

But, if f is strictly positive, (4.20) implies that it is impossible for both u_{xx} and u_{yy} to be non positive (they sum to f). Hence u cannot have an interior maximum within D, so it must attain its maximum value on the boundary ∂D.

Suppose now that we only have $f \geq 0$ in D. We perturb u to get a function v which satisfies Poisson's equation with the right hand side strictly positive, so we can apply the first part of the proof.

Consider the function
$$v(x,y) = u(x,y) + \frac{\epsilon}{4}(x^2 + y^2),$$

where ϵ is a positive constant.

Then
$$v_{xx} + v_{yy} = f + \epsilon > 0$$

in D. So using the result just proved, v attains its maximum value on ∂D.

Now, suppose that the maximum value of u on ∂D is M and the maximum value of $(x^2 + y^2)$ on ∂D is R^2, then the maximum value of v on ∂D (and thus throughout D) is $M + (\epsilon/4)R^2$. In other words, the inequality

$$u + \frac{\epsilon}{4}(x^2 + y^2) = v \leq M + \frac{\epsilon}{4}R^2$$

holds for all $(x, y) \in D$. Letting $\epsilon \to 0$, we see that $u \leq M$ throughout D, i.e. that u attains its maximum value on ∂D. $\qquad\square$

It obviously follows (by using the above result with u replaced by $-u$) that, if $f \leq 0$ in D, then u attains its minimum value on ∂D.

In the case $f = 0$, u therefore attains both its maximum and minimum values on ∂D. This is an important property of *Laplace's equation*.

Remark: (a) The result also holds if f is also a function of u, provided $f(x, y, u) \geq 0$ in D for all u.

(b) Note that if $f > 0$ then the maximum of u will be attained only on the boundary, whereas if $f \leq 0$ the maximum may also be attained within the interior of D as well as on the boundary.

Corollary 4.2. *(a) Consider the Dirichlet problem (4.18), (4.19). Then if the solution exists, it is unique.*

(b) The Dirichlet problem (4.18), (4.19) has continuous dependence on the data.

Proof: (a) Suppose that u_1, u_2 are two solutions, so $u = u_1 - u_2$ satisfies

$$\begin{align}
u_{xx} + u_{yy} &= 0 \quad \text{in } D \tag{4.21}\\
u &= 0 \quad \text{on } \partial D. \tag{4.22}
\end{align}$$

By (4.21) the maximum and minimum of u occur on ∂D, so by (4.22) $u \leq 0$ and $u \geq 0$ in D. Thus $u = 0$ in D as required.

(b) We have to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that if u_i, $i = 1, 2$ are solutions with boundary data g_i, then

$$\sup_{(x,y)\in\partial D} |g_1(x, y) - g_2(x, y)| < \delta \quad \Rightarrow \quad \sup_{(x,y)\in D} |u_1(x, y) - u_2(x, y)| < \epsilon.$$

By linearity $u = u_1 - u_2$ satisfies

$$\begin{align}
u_{xx} + u_{yy} &= 0 \text{ in } D \tag{4.23}\\
u &= g_1 - g_2, \text{ on } \partial D. \tag{4.24}
\end{align}$$

We now apply the maximum principle to see that $u \leq \max_{(x,y)\in\partial D}(g_1 - g_2)$, and applying the same result to $-u$, $-u \leq \max_{(x,y)\in\partial D} -(g_1 - g_2)$.

Hence in D

$$|u_1 - u_2| \leq \max_{(x,y) \in \partial D} |g_1 - g_2|, \qquad (4.25)$$

so we may take $\delta = \epsilon$. $\qquad\square$

4.4.2 The heat equation

In parabolic PDEs it is usually the case that one independent variable represents time, so we now use x and t as independent variables instead of x and y. The normal form for second-order parabolic equations is

$$u_{xx} = F(x, t, u, u_t, u_x)$$

and specific examples include the inhomogeneous heat equation, often called the diffusion equation:

$$u_t = u_{xx} + f(x, t).$$

and the reaction-diffusion equation

$$u_t = u_{xx} + f(x, t, u),$$

Well posed boundary data: Typical boundary data for a diffusion equation are to give an initial condition for u at $t = 0$ and one boundary condition on each of two curves C_1 and C_2 in the (x, t)-plane that do not meet and are nowhere parallel to the x-axis.

For example: The inhomogeneous heat equation

$$u_t = u_{xx} + f(x, t)$$

is a simple model for the temperature $u(x, t)$ in a uniform bar of conductive material, with heat source $f(x, t)$, where x is position and t is time. Suppose the bar is of length L, its initial temperature is given via $u_0(x)$, and its ends are kept at zero temperature. Then the initial and boundary conditions are

$$u = u_0(x) \text{ at } t = 0, \quad u = 0 \text{ at } x = 0; \ u = 0 \text{ at } x = L.$$

If, instead of being held at constant temperature, an end is insulated, then the Dirichlet boundary condition, $u = 0$, there is replaced by the Neumann boundary condition, $u_x = 0$. Alternatively , the boundary conditions at $x = 0$ and $x = L$ may, in general, be replaced by conditions at moving boundaries, say $x = x_1(t)$ and $x = x_2(t)$.

Theorem 4.3. *(The Maximum principle for the heat equation) Suppose that $u(x, t)$ satisfies the inhomogeneous heat equation*

$$u_t - u_{xx} = f(x, t) \tag{4.26}$$

in a region D_τ bounded by the lines $t = 0$, $t = \tau > 0$, and two non-intersecting smooth curves C_1 and C_2 that are nowhere parallel to the x-axis. Suppose also that $f \leq 0$ in D_τ. Then u takes its maximum value either on $t = 0$ or on one of the curves C_1 or C_2 .

Proof:

The proof is similar to that for Poisson's equation.

Suppose first that $f < 0$ in D_τ. At an internal maximum inside D_τ, u must satisfy

$$u_x = u_t = 0 \ u_{xx} \leq 0, \ (u_{tt} \leq 0).$$

On the other hand, if u has a maximum at a point on $t = \tau$, then there it must satisfy

$$u_x = 0, \ u_t \geq 0, u_{xx} \leq 0.$$

With f assumed to be strictly negative, both of these lead to contradictions, and it follows that u must take its maximum value somewhere on ∂D_τ but not on $t = \tau$. We are done.

Suppose now that $f \leq 0$, then define

$$v(x,t) = u(x,t) + \frac{\epsilon}{2}x^2,$$

where ϵ is a positive constant. Then v satisfies

$$v_t - v_{xx} = f(x,t) - \epsilon < 0$$

in D_τ. So by the earlier step v takes its maximum value on ∂D_τ but not on $t = \tau$.

Now if the maximum value of u over these three portions of ∂D_τ is M, and the maximum value of $|x|$ on C_1 and C_2 is L, then

$$u \leq v \leq \frac{L^2 \epsilon}{2} + M.$$

Now we let $\epsilon \to 0$ and conclude that $u \leq M$, i.e. u takes its maximum value on ∂D_τ but not on $t = \tau$ $\qquad\qquad\qquad \square$

If $f \geq 0$ in D_τ, then a similar argument shows that u attains its minimum value on ∂D_τ but not on $t = \tau$. Thus, for the homogeneous equation (the heat equation) u attains both its maximum and its minimum values on ∂D_τ but not on $t = \tau$.

Remark: (a) Again the result also holds if f is also a function of u, provided $f(x,t,u) \leq 0$ in D_τ for all u.

(b) Physical interpretation: In a rod with no heat sources the hottest and the coldest spot will occur either initially or at an end. (Because heat flows from a hotter area to a colder area.)

Corollary 4.4. *Consider the IBVP consisting of (4.26) in D_τ with u given on $\partial D_\tau \backslash \{t = \tau\}$. Then if the solution exists, it is unique and depends continuously on the initial data.*

Proof: As for Poisson's equation (see problem sheet).

Sketch: Suppose that u_i $(i = 1, 2)$ are solutions of (4.26) in D_τ with data

$u_i = g_i$ on C_1, $u_i = h_i$ on C_2, $u_i(x, 0) = k_i(x)$ for x between C_1 and C_2 .

Let $u = u_1 - u_2$, $g = g_1 - g_2$, $h = h_1 - h_2$, $k = k_1 - k_2$. Then u satisfies (4.26) with $f = 0$, and $u = g$ on C_1, $u = h$ on C_2, $u(x, 0) = k(x)$ for x between C_1 and C_2 . Thus u and $-u$ take their maximum values on $\partial D_\tau \backslash \{t = \tau\}$. Hence

$$\sup_{D_\tau} u \leq \max\{\sup g, \sup h, \sup k\}$$

and
$$\sup_{D_\tau} -u \leq \max\{\sup -g, \sup -h, \sup -k\}.$$

So
$$\sup_{D_\tau} |u| \leq \max\{\sup |g|, \sup |h|, \sup |k|\},$$

and continuous dependence follows. In particular, if $g = h = k = 0$ then $u = 0$ and uniqueness follows.

5 Where does this course lead?

The course leads to DEs2, where, among other topics, boundary value problems for ODEs are discussed. Further discussion of differential equations comes in the Part B courses 'Non-linear Systems' and 'Applied Partial Differential Equations'. The use of abstract methods such as the Contraction Mapping Theorem to investigate the solutions of differential equations is taken further in various C4 courses, which require some knowledge of Banach and Hilbert spaces.

The techniques taught in DEs1 and DEs2 will be useful in various applied maths courses such as the Part A short course 'Modelling in Mathematical Biology' and the Part B courses 'Mathematical Ecology and Biology', 'Viscous Flow' and 'Waves and Compressible Flow'.

5.1 Section 1

Fixed point results such as the CMT provide very powerful methods for proving existence of solution for ODEs and PDEs. For PDEs we have to work in Banach spaces of functions rather than $\mathbb{R}^n$. For example for parabolic equations such as the reaction-diffusion equation:

$$u_t = u_{xx} + f(t, u),$$

with suitable boundary data, our proof of Picard's theorem can be extended to prove local existence (in t), provided f is continuous and is Lipschitz in u, where for each time t the solution u lives within a Banach space of x-dependent functions (which is infinite dimensional space) rather than the Euclidean n-dimensional space that we considered previously for ODEs. The technical details of this require methods from Functional Analysis, but the basic ideas are just the same - there is just more to do, because more can go wrong!

There is more about fixed point methods in the Part C course 'Fixed Point Methods for Nonlinear PDEs'

An example of a reaction diffusion equations which occurs in applications is Fisher's equation:

$$u_t = u_{xx} + u(1 - u),$$

Another example of a second order parabolic equation is the Black-Scholes equation in mathematical finance.

5.2 Section 2

Phase plane analysis is a very important tool in mathematical modelling. It will be used, for example, in the Part A short course 'Modelling in Mathematical Biology'

and the Part B course, 'Mathematical Ecology and Biology'.

The theory will be taken further in the Part B course, 'Nonlinear Systems'.

5.3 Section 3

We have considered only semi-linear first order PDEs. Similar methods can be extended to quasi-linear and fully non-linear equations. In these cases the characteristic equations are generally more difficult to solve. Such equations allow for the formation of shocks - see Part B 'Applied PDEs'.

These first order PDEs model many physical processes, including particularly conservation laws, and will appear in many of the modelling courses in Part B and beyond.

There are other methods for producing explicit solutions of PDEs:

Transform methods are very useful for linear PDEs- see the Part A short course in HT.

Similarity solutions can be used for both linear or non-linear PDEs - see Part B course 'Applied PDEs' - and involve reducing the PDE to an ODE in a 'similarity variable' involving both independent variables in the PDE.

5.4 Section 4

As we have already observed, proving the existence of solution, even of semi-linear second order PDEs is challenging. The different types of equation demand different approaches. For example:

Semi-linear hyperbolic equations with the solution u and its normal derivative prescribed on a given initial curve: One method to prove existence of solutions proceeds by showing that on initial curves other than characteristic curves we can find all the derivatives of u and that if all coefficients etc in the equation are very smooth (analytic) the solution is given by a power series near the initial curve. This is a version of the Cauchy-Kowalevski theorem.

Hyperbolic equations are dealt with further in the Part C course 'Hyperbolic equations'.

Elliptic equations are treated in the Part C course 'Functional analytic methods for PDEs'.

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